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Hamilton-Jacobi Equations on an almost Kähler Model of a Cartan Space

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Özet

As known, Hamiltonian models arise to be a very important tool in modern geometry. Because they present a simple method to describe the different model for mechanical systems. Therefore in this paper we obtain Hamilton-Jacobi Equations on an almost Kähler model of a Cartan manifold by using this model. In the conclusion we discuss some results about related mechanical system.

Bir Cartan Uzayının Hemen Hemen Kähler Modeli Üzerindeki Hamilton-Jakobi Denklemleri

Anahtar kelimeler

Cartan manifoldu;
Kähler Manifoldu;
Hamilton Mekaniği;
Hamilton-Jakobi
Denklemleri

Abstract

Bilindiği üzere Hamilton modelleri modern geometride giderek önem kazanmaktadır. Çünkü bu modeller mekanik sistemlerin farklı modellerini tasvir etmek için daha kolay bir yöntem sağlamaktadır. Bundan dolayı çalışmada bahsi geçen model kullanılarak Bir Cartan Uzayının Hemen Hemen Kähler modeli üzerinde Hamilton-Jakobi hareket denklemlerini elde edilmiştir. Sonuç kısmında ise ilgili mekanik sistemler üzerinde elde edilen bulgular tartışılmıştır.

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1. Introduction

Hamiltonian mechanics uses the fundamental structures of modern differential geometry. Because a suitable vector field on a cotangent space which is the phase space of momentum of a given configuration space explains the dynamics of Hamiltonians. These dynamics formalisms and Hamilton equations are set as follows:

Let M is m -dimensional configuration manifold and its cotangent bundle T^*M . If $H: T^*M \rightarrow \mathbb{R}$ is a regular Hamiltonian function then there is a unique vector field X_H on T^*M such that Hamiltonian dynamical formalism

$$i_{X_H}\phi = dH \quad (1)$$

where ϕ is the canonical symplectic form so that $\phi = -d\lambda$, $\lambda = J^*(\omega)$, J^* a dual of J , ω a 1-form on T^*M . Further, the geodesics of the manifold M can be found by solving Hamiltonian equations shown by

$$\frac{dx^i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial x^i} \quad (2)$$

where (x^i) and (x^i, p_i) , $1 \leq i \leq m$, are coordinates of M and T^*M respectively. Furthermore the triple (T^*M, ϕ, X_H) is called Hamiltonian system on T^*M (de Leon and Rodrigues, 1989; Miron et al, 2001). It is possible to find many studies about Hamiltonian mechanics, dynamics and equations in articles (Tekkoyun and Yaylı, 2011; Tekkoyun and Celik 2013; Klein, 1963) and in books (de Leon and Rodrigues, 1985).

Hamilton-Jacobi equations (HJE) have also been an interesting area for mathematicians and physicists. Since they have many applications in physics, the solutions of these equations are investigated in many papers.

Crandall et al. studied viscosity solutions of HJE, also Xuehong Zhu presented viscosity solutions of HJE on Riemannian manifolds which make the probabilistic interpretation for nonlinear PDEs on Riemannian manifolds (Crandall and Lions, 1983; Zhu, 2014) P. L. Lions, G. Papanicolaou and S. R. S. Varadhan are concerned with homogenization of HJE in their paper (Lions et al., unpublished). P. Jameson Graber showed the optimal control of solutions of first order Hamilton-Jacobi equations, where the Hamiltonian is convex with linear growth (Graber, 2014).

In this study, we set Hamilton-Jacobi equations on an almost Kähler model of a Cartan manifold. In the

conclusion we say some results about related mechanical system and equations.

2. Cartan Structure and Almost Kähler Model of a Cartan Space

In this section we remind of some structures given by Riron and Rodrigues (2001). Let M a smooth manifold of real dimension m and $K: T^*M \rightarrow [0, \infty]$ is a real function. Then the pair $(C^n, K(x, p))$ is a Cartan space if the following axioms hold:

- i) K is differentiable on $T^*M \setminus \{0\}$ for $\{0\} = \{(x, 0) | x \in M\}$
- ii) K is positively 1-homogeneous on the fibres of cotangent bundle T^*M that is $K(x, \lambda p) = \lambda K(x, p)$ for all $\lambda > 0$
- iii) The Hessian of K^2 with elements

$$g^{ij}(x, p) = \frac{1}{2} \frac{\partial^2 K^2}{\partial p_i \partial p_j} \tag{3}$$

is positively defined on $T^*M \setminus \{0\}$.

It can be seen that g^{ij} is symmetric and nonsingular distinguished tensor field (d-tensor field) and contravariant of order 2 on the manifold

$T^*M \setminus \{0\}$. The function $K(x, p)$ is called fundamental function and the d-tensor field g^{ij} is called fundamental (or metric) tensor of the Cartan Space $C^n = (M, K(x, p))$. A Cartan space $C^n = (M, K)$ can be thought as an almost Kähler space on the manifold $T^*M \setminus \{0\}$, called the geometrical model of the Cartan space. Considering nonlinear connection N_i^j on T^*M with $N: u \in T^*M \mapsto N_u \subset T_u T^*M$ then we can write $T_u T^*M = H_u T^*M \oplus V_u T^*M$ for all $u \in T^*M$ where H and V are horizontal and vertical distributions respectively. Afterwards it follows that

$$\left(\frac{\partial}{\partial x^i}\right)^H = \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^j(x, p) \frac{\partial}{\partial p_j} \tag{4}$$

$$\left(\frac{\partial}{\partial p_i}\right)^V = \frac{\partial}{\partial p_i} \tag{5}$$

Moreover $\left(\frac{\delta}{\delta x^i}\right), i = 1, \dots, m$ is a local base in HTM , $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial p_i}\right) i = 1, \dots, m$ is a local base adapted to the HTM and VTM . And also $(dx^i, \delta p_i) i = 1, \dots, m$ is the dual base of $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial p_i}\right) i = 1, \dots, m$ where $\delta p_i = dx^i + N_i^j(x, p) dp_j \in HT^*M$. Suppose that the d-tensor metric $g^{ij}(x, p)$ and the symmetric nonlinear connection N are given then we can define almost complex structure F as follows:

$$F\left(\frac{\partial}{\partial x^i}\right) = -g_{ij} \frac{\partial}{\partial p_j} \quad F\left(\frac{\partial}{\partial p_i}\right) = g^{ij} \frac{\delta}{\delta x^i} \tag{6}$$

Then the affect of F on the base $(dx^i, \delta p_i)$ as follows:

$$F^*(dx^i) = -g_{ij} \delta p_j \quad F^*(\delta p_i) = g^{ij} dx^j \tag{7}$$

The N-lift of the tensor field g^{ij} of C^n can be introduced as follows:

$$G = g_{ij} dx^i \otimes dx^j + g^{ij} \delta p_i \otimes \delta p_j \tag{8}$$

Consequently, G is a Riemann metric on $T^*M \setminus \{0\}$ determined only by the fundamental function K of the space C^n and the horizontal and vertical distributions are orthogonal with respect to it.

Theorem 1. (Miron et al, 2001)

i) The pair (G, F^*) is an almost Hermitian structure on T^*M

ii) The almost symplectic 2-form associated to the almost Hermitian structure (G, F^*) is

$$\theta = \delta p_i \wedge dx^i \quad (9)$$

iii) The space $H^{2m} = (T^*M \setminus \{0\}, G, F^*)$ is an almost Kahler space, constructed only by means of fundamental function K of the space C^n .

Definition 1.: The space $H^{2m} = (T^*M \setminus \{0\}, G, F^*)$ is called almost Kahler model of the space C^n

3. Hamilton-Jacobi Equations

Here we present Hamilton-Jacobi equations and Hamiltonian mechanical systems for quantum and classical mechanics constructed on the almost Kahler model $H^{2m} = (T^*M \setminus \{0\}, G, F^*)$ of the space C^n

Theorem 2. (Miron et al, 2001)

There exists a unique vector field $X_{K^2} \in \chi(T^*M \setminus \{0\})$ with the property

$$i_{X_{K^2}} \phi = -dK^2 \quad (10)$$

for ant Cartan space. This equation is called Hamilton-Jacobi formalism.

Now, firstly let set a 1-form

$$\begin{aligned} g_{ij} dx^i \otimes dx^j + g^{ij} \delta p_i \otimes \delta p_j \\ \omega = \frac{1}{2} g^{ij} x^i dx^i + \frac{1}{2} g_{ij} p_i \delta p_i \end{aligned} \quad (11)$$

Then we have the Lioville form

$$\lambda = F^*(\omega) = -\frac{1}{2} x^i \delta p_i + \frac{1}{2} p_i dx^i \quad (12)$$

and the closed form

$$\phi = -d\lambda = dx^i \wedge \delta p_i \quad (13)$$

Set Hamiltonian vector field as follows:

$$X_{K^2} = X^i \frac{\partial}{\partial x^i} + Y^i \frac{\delta}{\delta p_i} \quad (14)$$

Then we find

$$i_{X_{K^2}} \phi = \phi(X_{K^2}) = X^i \delta p_i - Y^i dx^i \quad (15)$$

and

$$-dK^2 = -\frac{\delta K^2}{\delta x^i} dx^i - \frac{\partial K^2}{\partial p_i} \delta p_i \quad (16)$$

By means of (10), then the Hamiltonian vector field X_{K^2} is found as follows:

$$X_{K^2} = -\frac{\partial K^2}{\partial p_i} \frac{\partial}{\partial x^i} + \frac{\delta K^2}{\delta x^i} \frac{\delta}{\delta p_i} \quad (17)$$

Assume that a curve

$$\alpha: I \subset \mathbb{R} \rightarrow T^*M \setminus \{0\} \quad (18)$$

be an integral curve of the Hamiltonian vector field X_{K^2} , i.e.,

$$X_{K^2}(\alpha(t)) = \dot{\alpha}(t) \quad (19)$$

In the local coordinates, it is obtained that

$$\alpha(t) = (x^i, p_i) \quad (20)$$

and

$$\dot{\alpha}(t) = \frac{dx^i}{dt} \frac{\partial}{\partial x^i} + \frac{dp_i}{dt} \quad (21)$$

Considering (19), (17) and (21) we find

$$\frac{dx^i}{dt} = -\frac{\partial K^2}{\partial p_i}, \quad \frac{dp_i}{dt} = \frac{\delta K^2}{\delta x^i} \quad (22)$$

Hence the equations in (22) are named Hamilton-Jacobi equations on the almost Kahler model H^{2m} of the manifold C^n . And then the triple (H^{2m}, ϕ, X_{K^2}) is said to be a Hamiltonian mechanical system.

4. Conclusion

In this study, Hamiltonian mechanical system has intrinsically been described on the almost Kähler model $H^{2m} = (T^*M \setminus \{0\}, G, F^*)$ of the Cartan space C^n . The paths of the Hamilton vector field X_{K^2} on the almost Kähler model H^{2m}

are the solutions of Hamilton -Jacobi equations raised in (22).

Nowadays, as known Hamiltonian models arise to be a very important tool since they present a simple method to describe the model for mechanical systems. One can be proved that the obtained Hamilton-Jacobi equations are very important to explain situations as viscosity, probabilistic interpretation, optimal control, homogenization. Therefore, the found equations are only considered to be a first step to realize how a Hamilton-Jacobi equations have been used in solving problems in different mathematical and physical areas.

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