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Bertrand Partner D-Curves in the Euclidean 3-space E^3

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Abstract

In this paper, we consider the idea of Bertrand partner curves for curves lying on surfaces and by considering the Darboux frames of surface curves, we call these curves as Bertrand partner D-curves and give the characterizations for these curves by means of the geodesic curvatures, the normal curvatures and the geodesic torsions of these associated curves.

E^3 Öklid 3-Uzayında Bertrand Partner D-Eğrileri

Anahtar kelimeler
Bertrand Partner Eğrileri; Darboux Çatısı; Geodezik; Asli Doğrultu Eğrisi; Asimptotic Eğri.

Özet

Bu çalışmada, Bertrand partner eğrileri düşüncesi yüzey üzerinde yatan eğriler için ele alınmış ve yüzey eğrilerinin Darboux çatıları dikkate alınarak bu eğriler Bertrand partner D-eğrileri olarak adlandırılmıştır. Bu eğrilerin karakterizasyonları, bağlantılı eğrilerin geodezik eğriliklerine, normal eğriliklerine ve geodezik burulmalarına göre verilmiştir.

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1. Introduction

Bertrand partner curves are one of the associated curve pairs for which at the corresponding points of the curves one of the Frenet vectors of a curve coincides with the one of the Frenet vectors of the other curve. Bertrand partner curves are very interesting and an important problem of the fundamental theory and the characterizations of space curves and are characterized as a kind of corresponding relation between two curves such that the curves have the common principal normal, i.e., the Bertrand curve is a curve which shares the principal normal line with another curve. A Bertrand curve α is characterized by the equality $\lambda\kappa(s)+\mu\tau(s)=1$, where $\lambda,\ \mu$ are constants and $\kappa(s),\ \tau(s)$ are the curvature and the torsion of the curve, respectively (Bertrand, 1850). These curves have an important role in the theory of curves and surfaces. Hereby, from the past to today, a lot of mathematicians have studied on Bertrand curves in different areas (Burke, 1960; Görgülü and Özdamar, 1986; Struik, 1988; Whittemore, 1940). Moreover, these curves are related to some other special curves and surfaces. Izumiya and Takeuchi (2003) have studied cylindrical helices and Bertrand curves from the view point as curves on ruled surfaces. They have shown that cylindrical helices can be constructed from plane curves and Bertrand curves can be constructed from spherical curves. Also, they have studied generic properties of cylindrical helices and Bertrand curves as applications of singularity theory for plane curves and spherical curves (Izumiya and Takeuchi, 2002). Moreover, Bertrand partner curves have been defined in the three-dimensional sphere S^3 and another definition for space curves to be Bertrand curves immersed in S^3 have been introduced by Lucas and Ortega-Yagües, (2012). In the same paper, the authors have obtained that a curve α with curvatures $\kappa_\alpha,\ \tau_\alpha$ immersed in S^3 is a Bertrand curve if and only if either $\tau_\alpha \equiv 0$ and α is a curve in some unit two-dimensional sphere $S^2(1)$ or there exit two constants $\lambda \neq 0,\ \mu$ such that $\lambda\kappa_\alpha + \mu\tau_\alpha = 1$.

In this paper, we consider the notion of the Bertrand curve for curves lying on the surfaces. We call these new associated curves as Bertrand partner D-curves and by using the Darboux frames of the curves we give definition and characterizations of these curves. We obtained that two curves are Bertrand partner D-curves if and only if their curvatures satisfy the equality given in Theorem 3.1. Later, we obtain some special cases given in Theorem 3.2 and Theorem 3.3.

2. Darboux Frame of a Curve Lying on a Surface

Let $S = S(u,v)$ be an oriented surface in the 3-dimensional Euclidean space E^3 and let consider a curve $x(s)$ lying fully on S where $(u,v) \in U \subset IR^2$, U is an open set and s is the arc length parameter of curve $x(s)$. Since the curve $x(s)$ is also in space, there exists a Frenet frame $\{T, N, B\}$ along the curve where T is unit tangent vector, N is principal normal vector and B is binormal vector, respectively. The Frenet equations of the curve $x(s)$ is given by

$$\dot{T} = \kappa N$$
$$\dot{N} = -\kappa T + \tau B$$
$$\dot{B} = -\tau N$$

where κ and τ are curvature and torsion of the curve $x(s)$, respectively, and "dot" shows the derivative with respect to arc length parameter s.

Since the curve $x(s)$ also lies on the surface S there exists another frame along $x(s)$ which is called Darboux frame and denoted by $\{T, g, n\}$. In this frame T is the unit tangent of the curve, n is the unit normal of the surface S along $x(s)$ and g is a unit vector defined by $g = n \times T$ where $\times$ denotes the vector product in E^3. Since the unit tangent T is common in both Frenet frame and Darboux frame, the vectors $N,\ B,\ g$ and n lie on the same plane. So that the relations between these frames can be given as follows

$$\begin{bmatrix} T \\ g \\ n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

where φ is the angle between the vectors g and N. The derivative formulas of the Darboux frame is

$$\begin{bmatrix} \dot{T} \\ \dot{g} \\ \dot{n} \end{bmatrix} = \begin{bmatrix} 0 & k_g & k_n \\ -k_g & 0 & \tau_g \\ -k_n & -\tau_g & 0 \end{bmatrix} \begin{bmatrix} T \\ g \\ n \end{bmatrix} \qquad (1)$$

where $k_g,\ k_n$ and τ_g are called the geodesic curvature, the normal curvature and the geodesic torsion, respectively. Here and in the following, we use "dot" to denote the derivative with respect to the arc length parameter of a curve.

The relations between geodesic curvature, normal curvature, geodesic torsion and κ, τ are given as follows

$$k_g = \kappa\cos\varphi\,,\quad k_n = \kappa\sin\varphi\,,\quad \tau_g = \tau + \frac{d\varphi}{ds}. \quad (2)$$

Furthermore, the geodesic curvature k_g and geodesic torsion τ_g of the curve $x(s)$ can be calculated as follows

$$k_g = \left\langle \frac{dx}{ds}, \frac{d^2x}{ds^2}\times n \right\rangle,\quad \tau_g = \left\langle \frac{dx}{ds}, n\times\frac{dn}{ds} \right\rangle \quad (3)$$

where $\langle , \rangle$ denotes the inner product in E^3. In the differential geometry of surfaces, for a surface curve $x(s)$ the followings are well-known:

i) $x(s)$ is a geodesic curve $\Leftrightarrow k_g = 0$,

ii) $x(s)$ is an asymptotic line $\Leftrightarrow k_n = 0$,

iii) $x(s)$ is a principal line $\Leftrightarrow \tau_g = 0$.

(See O'Neill, (1966) and Sturik, (1988) for details).

3. Bertrand Partner D-Curves in the Euclidean 3-space E^3

In this section, by considering the Darboux frame, we define Bertrand D-curves and give the characterizations of these curves.

Definition 3.1. Let S and S_1 be oriented surfaces in E^3 and let consider the unit speed curves $x(s)$ and $x_1(s_1)$ lying fully on S and S_1, respectively. Denote the Darboux frames of $x(s)$ and $x_1(s_1)$ by $\{T, g, n\}$ and $\{T_1, g_1, n_1\}$, respectively. If there exists a corresponding relationship between the curves x and x_1 such that, at the corresponding points of the curves, direction of the vector g coincides with direction of the vector g_1, then x is called a Bertrand D-curve, and x_1 is called a Bertrand partner D-curve of x. Then, the pair $\{x, x_1\}$ is said to be a Bertrand D-pair.

Theorem 3.1. *Let S be an oriented surface and $x(s)$ be a curve lying on S in E^3 with arc length parameter s. If S_1 is another oriented surface and $x_1(s_1)$ is a curve with arc length parameter s_1 lying on S_1, then $x_1(s_1)$ is Bertrand partner D-curve of $x(s)$ if and only if the normal curvature k_n of $x(s)$ and the geodesic curvature k_{g_1}, the normal curvature k_{n_1} and the geodesic torsion τ_{g_1} of $x_1(s_1)$ satisfy the following equation,*

$$-\lambda\dot{\tau}_{g_1} = \left(\frac{(1-\lambda k_{g_1})^2 + \lambda^2\tau_{g_1}^2}{(1-\lambda k_{g_1})}\right)\left(k_{n_1} - k_n\frac{1-\lambda k_{g_1}}{\cos\theta}\right) + \frac{\lambda^2\tau_{g_1}\dot{k}_{g_1}}{1-\lambda k_{g_1}}$$

for some nonzero constants λ, where θ is the angle between the tangent vectors T and T_1 at the corresponding points of x and x_1.

Proof: Let $x(s)$ and $x_1(s_1)$ be Bertrand D-curves with Darboux frames $\{T, g, n\}$ and $\{T_1, g_1, n_1\}$, respectively. Then by the definition we can write

$$x(s_1) = x_1(s_1) + \lambda(s_1)g_1(s_1), \quad (4)$$

for some function $\lambda(s_1)$. By taking derivative of (4) with respect to s_1 and applying the Darboux formulas (1) we have

$$T\frac{ds}{ds_1} = (1-\lambda k_{g_1})T_1 + \dot{\lambda}g_1 + \lambda\tau_{g_1}n_1. \quad (5)$$

Since the direction of g_1 coincides with the direction of g, i.e., the tangent vector T of the curve lies on the plane spanned by the vectors T_1 and n_1, we get

$$\dot{\lambda}(s_1) = 0.$$

This means that λ is a non-zero constant. Thus, the equality (5) can be written as follows

$$T\frac{ds}{ds_1}=(1-\lambda k_{g_1})T_1+\lambda\tau_{g_1}n_1. \qquad (6)$$

Furthermore, we have

$$T=\cos\theta T_1-\sin\theta n_1, \qquad (7)$$

where θ is the angle between the tangent vectors T and T_1 at the corresponding points of x and x_1. By differentiating this last equation with respect to s_1, we get

$$\begin{aligned}(k_g g+k_n n)\frac{ds}{ds_1}&=(-\dot{\theta}+k_{n_1})\sin\theta T_1\\&+(k_{g_1}\cos\theta+\tau_{g_1}\sin\theta)g_1\\&+(-\dot{\theta}+k_{n_1})\cos\theta n_1\end{aligned} \qquad (8)$$

From this equation and the fact that

$$n=\sin\theta T_1+\cos\theta n_1, \qquad (9)$$

we get

$$\begin{aligned}&(k_n\sin\theta T_1+k_g g+k_n\cos\theta n_1)\frac{ds}{ds_1}\\&=(-\dot{\theta}+k_{n_1})\sin\theta T_1+(k_{g_1}\cos\theta+\tau_{g_1}\sin\theta)g_1\\&+(-\dot{\theta}+k_{n_1})\cos\theta n_1\end{aligned} \qquad (10)$$

Since the direction of g_1 is coincident with direction of g we have

$$\dot{\theta}=k_{n_1}-k_n\frac{ds}{ds_1}. \qquad (11)$$

From (6) and (7) and notice that T_1 is orthogonal to g_1 we obtain

$$\frac{ds}{ds_1}=\frac{1-\lambda k_{g_1}}{\cos\theta}=\frac{-\lambda\tau_{g_1}}{\sin\theta}. \qquad (12)$$

Equality (12) gives us

$$\tan\theta=\frac{-\lambda\tau_{g_1}}{1-\lambda k_{g_1}}. \qquad (13)$$

By taking the derivative of this equation and applying (11) we get

$$\begin{aligned}-\lambda\dot{\tau}_{g_1}&=\left(\frac{(1-\lambda k_{g_1})^2+\lambda^2\tau_{g_1}^2}{(1-\lambda k_{g_1})}\right)\left(k_{n_1}-k_n\frac{1-\lambda k_{g_1}}{\cos\theta}\right)\\&+\frac{\lambda^2\tau_{g_1}\dot{k}_{g_1}}{1-\lambda k_{g_1}}\end{aligned} \qquad (14)$$

that is desired.

Conversely, assume that the equation (14) holds for some non-zero constants λ. Then by using (12) and (13), (14) gives us

$$\begin{aligned}k_n\left(\frac{ds}{ds_1}\right)^3&=\lambda\dot{\tau}_{g_1}(1-\lambda k_{g_1})+\lambda^2\tau_{g_1}\dot{k}_{g_1}\\&+\left((1-\lambda k_{g_1})^2+\lambda^2\tau_{g_1}^2\right)k_{n_1}\end{aligned} \qquad (15)$$

Let define a curve

$$x(s_1)=x_1(s_1)+\lambda g_1(s_1). \qquad (16)$$

We will prove that x is a Bertrand D-curve and x_1 is the Bertrand partner D-curve of x. By taking the derivative of (16) with respect to s_1 twice, we get

$$T\frac{ds}{ds_1}=(1-\lambda k_{g_1})T_1+\lambda\tau_{g_1}n_1, \qquad (17)$$

and

$$\begin{aligned}(k_g g+k_n n)\left(\frac{ds}{ds_1}\right)^2+T\frac{d^2s}{ds_1^2}&\\=-\lambda(\dot{k}_{g_1}+\tau_{g_1}k_{n_1})T_1&\\+\left((1-\lambda k_{g_1})k_{g_1}-\lambda\tau_{g_1}^2\right)g_1&\\+\left((1-\lambda k_{g_1})k_{n_1}+\lambda\dot{\tau}_{g_1}\right)n_1&\end{aligned} \qquad (18)$$

respectively. Taking the cross product of (17) with (18) we have

$$\begin{aligned}[k_g n - k_n g]\left(\frac{ds}{ds_1}\right)^3 &= \left[-\lambda\tau_{g_1}k_{g_1}(1-\lambda k_{g_1})+\lambda^2\tau_{g_1}^3\right]T\\ &-\left[(1-\lambda k_{g_1})^2 k_{n_1}+\lambda\dot{\tau}_{g_1}(1-\lambda k_{g_1})+\lambda^2\tau_{g_1}\dot{k}_{g_1}+\lambda^2\tau_{g_1}^2 k_{n_1}\right]g_1 \\ &+\left[k_{g_1}(1-\lambda k_{g_1})^2-\lambda\tau_{g_1}^2(1-\lambda k_{g_1})\right]n_1.\end{aligned} \tag{19}$$

By substituting (15) in (19) we get

$$\begin{aligned}[k_g n - k_n g]\left(\frac{ds}{ds_1}\right)^3 &= \left(-\lambda\tau_{g_1}k_{g_1}(1-\lambda k_{g_1})+\lambda^2\tau_{g_1}^3\right)T_1\\ &-k_g\left(\frac{ds}{ds_1}\right)^3 g_1+\left(k_{g_1}(1-\lambda k_{g_1})^2-\lambda\tau_{g_1}^2(1-\lambda k_{g_1})\right)n_1.\end{aligned} \tag{20}$$

Taking the cross product of (17) with (20) we have

$$\begin{aligned}[-k_g g - k_n n]\left(\frac{ds}{ds_1}\right)^4 &= \lambda k_n\tau_{g_1}\left(\frac{ds}{ds_1}\right)^3 T_1\\ &+\left((1-\lambda k_{g_1})^2+\lambda^2\tau_{g_1}^2\right)\left(\lambda\tau_{g_1}^2-k_{g_1}(1-\lambda k_{g_1})\right)g_1\\ &-k_n(1-\lambda k_{g_1})\left(\frac{ds}{ds_1}\right)^3 n_1.\end{aligned} \tag{21}$$

From (20) and (21) we obtain

$$\begin{aligned}-(k_g^2+k_n^2)\left(\frac{ds}{ds_1}\right)^4 n &= \left[\lambda k_g k_{g_1}\tau_{g_1}(1-\lambda k_{g_1})\frac{ds}{ds_1}\right.\\ &\left.-\lambda^2 k_g\tau_{g_1}^3\frac{ds}{ds_1}+\lambda\tau_{g_1}k_n^2\left(\frac{ds}{ds_1}\right)^3\right]T_1\\ &+k_n\left(\frac{ds}{ds_1}\right)^2\left[k_g\left(\frac{ds}{ds_1}\right)^2+\lambda\tau_{g_1}^2-k_{g_1}(1-\lambda k_{g_1})\right]g_1\\ &+\left[-k_g k_{g_1}(1-\lambda k_{g_1})^2\frac{ds}{ds_1}+\lambda\tau_{g_1}^2 k_g(1-\lambda k_{g_1})\frac{ds}{ds_1}\right.\\ &\left.-k_n^2(1-\lambda k_{g_1})\left(\frac{ds}{ds_1}\right)^3\right]n_1.\end{aligned} \tag{22}$$

Furthermore, from (17) and (20) we get

$$\begin{cases}\left(\dfrac{ds}{ds_1}\right)^2=(1-\lambda k_{g_1})^2+\lambda^2\tau_{g_1}^2,\\[2ex] k_g\left(\dfrac{ds}{ds_1}\right)^2=k_{g_1}(1-\lambda k_{g_1})-\lambda\tau_{g_1}^2,\end{cases}$$

respectively. Substituting these two equalities in (22) gives us

$$\begin{aligned}-(k_g^2+k_n^2)\left(\frac{ds}{ds_1}\right)^4 n &= \left[\lambda k_g k_{g_1}\tau_{g_1}(1-\lambda k_{g_1})\frac{ds}{ds_1}\right.\\ &\left.-\lambda^2 k_g\tau_{g_1}^3\frac{ds}{ds_1}+\lambda\tau_{g_1}k_n^2\left(\frac{ds}{ds_1}\right)^3\right]T_1\\ &+\left[-k_g k_{g_1}(1-\lambda k_{g_1})^2\frac{ds}{ds_1}+\lambda\tau_{g_1}^2 k_g(1-\lambda k_{g_1})\frac{ds}{ds_1}\right.\\ &\left.-k_n^2(1-\lambda k_{g_1})\left(\frac{ds}{ds_1}\right)^3\right]n_1.\end{aligned}$$

Last equality and equality (17) shows that the vectors T and n lie on the plane $sp\{T_1, n_1\}$. So, at the corresponding points of the curves, direction of g coincides with direction of g_1, i.e, the curves x and x_1 are Bertrand D-pair curves.

From Theorem 3.1 we can give the followings special cases:

Assume that $x(s)$ is an asymptotic line. Then, from (14) we have the following results:

i) Consider that $x_1(s_1)$ is a geodesic curve. Then $x_1(s_1)$ is Bertrand partner D-curve of $x(s)$ if and only if the following equality holds,

$$\lambda\dot{\tau}_{g_1} = -k_{n_1}(1+\lambda^2\tau_{g_1}^2).$$

ii) Assume that $x_1(s_1)$ is also an asymptotic line. Then $x_1(s_1)$ is Bertrand partner D-curve of $x(s)$ if and only if the geodesic torsion τ_{g_1} of $x_1(s_1)$ satisfies the following equation,

$$\dot{\tau}_{g_1} = -\frac{\lambda\tau_{g_1}\dot{k}_{g_1}}{1-\lambda k_{g_1}}.$$

iii) If $x_1(s_1)$ is a principal line then $x_1(s_1)$ is Bertrand partner D-curve of $x(s)$ if and only if the geodesic curvature k_{g_1} and the geodesic torsion τ_{g_1} of $x_1(s_1)$ satisfy the following equality,

$$k_{n_1}(1-\lambda k_{g_1})=0.$$

Theorem 3.2. *Let the pair $\{x, x_1\}$ be a Bertrand D-pair. Then the relation between geodesic curvature k_g, geodesic torsion τ_g of $x(s)$ and the geodesic curvature k_{g_1}, the geodesic torsion τ_{g_1} of $x_1(s_1)$ is given as follows*

$$k_g - k_{g_1} = \lambda(k_g k_{g_1} - \tau_g \tau_{g_1}).$$

Proof: Let $x(s)$ be a Bertrand D-curve and $x_1(s_1)$ be a Bertrand partner D-curve of $x(s)$. Then from (16) we can write

$$x_1(s_1) = x(s_1) - \lambda g_1(s_1), \qquad (23)$$

for some constants λ. By differentiating (23) with respect to s_1 we have

$$T_1 = (1+\lambda k_g)\frac{ds}{ds_1}T - \lambda\tau_g\frac{ds}{ds_1}n. \qquad (24)$$

By the definition we have

$$T_1 = \cos\theta T + \sin\theta n. \qquad (25)$$

From (24) and (25) we obtain

$$\cos\theta = (1\lambda k_{g_1})\frac{ds_1}{ds}, \quad \sin\theta = -\lambda\tau_{g_1}\frac{ds_1}{ds}. \qquad (26)$$

Using (12) and (26) it is easily seen that

$$k_g - k_{g_1} = \lambda(k_g k_{g_1} - \tau_g \tau_{g_1}).$$

From Theorem 3.2, we obtain the following special cases:

Let the pair $\{x, x_1\}$ be a Bertrand D-pair. Then,

i) if one of the curves x and x_1 is a principal line, then the relation between the geodesic curvatures k_g and k_{g_1} is

$$k_g - k_{g_1} = \lambda k_g k_{g_1},$$

ii) if x_1 is a geodesic curve, then the geodesic curvature of the curve x is given by

$$k_g = -\lambda\tau_g\tau_{g_1},$$

iii) if x is a geodesic curve, then the geodesic curvature of the curve x_1 is given by

$$k_{g_1} = \lambda\tau_g\tau_{g_1},$$

Theorem 3.3. *Let $\{x, x_1\}$ be Bertrand D-pair. Then the following relations hold:*

i) $k_{n_1} = k_n\dfrac{ds}{ds_1} + \dfrac{d\theta}{ds_1}$

ii) $\tau_g\dfrac{ds}{ds_1} = k_{g_1}\sin\theta + \tau_{g_1}\cos\theta$

iii) $k_g\dfrac{ds}{ds_1} = k_{g_1}\cos\theta + \tau_{g_1}\sin\theta$

iv) $\tau_{g_1} = (k_g\sin\theta + \tau_g\cos\theta)\dfrac{ds}{ds_1}$

Proof: i) By differentiating the equation $\langle T, T_1\rangle = \cos\theta$ with respect to s_1 we have

$$\left\langle (k_g g + k_n n)\frac{ds}{ds_1}, T_1\right\rangle + \left\langle T, k_{g_1}g_1 + k_{n_1}n_1\right\rangle = -\sin\theta\frac{d\theta}{ds_1}$$

Using the fact that the direction of g_1 coincides with the direction of g and

$$T_1 = \cos\theta T + \sin\theta n, \quad n_1 = -\sin\theta T + \cos\theta n, \qquad (27)$$

we easily get that

$$k_{n_1} = k_n\frac{ds}{ds_1} + \frac{d\theta}{ds_1}.$$

ii) By differentiating the equation $\langle n, g_1\rangle = 0$ with respect to s_1 we have

$$\left\langle (-k_n T - \tau_g g)\frac{ds}{ds_1}, g_1 \right\rangle + \left\langle n, k_{g_1}T_1 + \tau_{g_1} n_1 \right\rangle = 0 .$$

By (27) we obtain

$$\tau_g \frac{ds}{ds_1} = k_{g_1}\sin\theta + \tau_{g_1}\cos\theta .$$

iii) By differentiating the equation $\langle T, g_1\rangle = 0$ with respect to s_1 we get

$$\left\langle (k_g g + k_n n)\frac{ds}{ds_1}, g_1 \right\rangle + \left\langle T, (-k_{g_1}T_1 + \tau_{g_1} n_1 \right\rangle = 0 .$$

From (27) it follows that

$$k_g \frac{ds}{ds_1} = k_{g_1}\cos\theta + \tau_{g_1}\sin\theta .$$

iv) Differentiating the equation $\langle n_1, g\rangle = 0$ with respect to s_1, we obtain

$$\left\langle -k_{n_1}T_1 - \tau_{g_1} g_1,\ g \right\rangle + \left\langle n_1,\ (-k_g T + \tau_g n)\frac{ds}{ds_1} \right\rangle = 0 ,$$

and using the fact that direction of g_1 coincides with the direction of g and

$$T = \cos\theta T_1 - \sin\theta n_1, \quad n = \sin\theta T_1 + \cos\theta n_1,$$

we get

$$\tau_{g_1} = (k_g \sin\theta + \tau_g\cos\theta)\frac{ds}{ds_1} .$$

Let now x be a Bertrand D-curve and x_1 be a Bertrand partner D-curve of x. From the first equation of (3) and by using the fact that $n_1 = -\sin\theta T + \cos\theta n$ we have

$$\begin{aligned} k_{g_1} &= \left[\left(1 + \lambda k_g\right)\cos\theta - \lambda\tau_g \sin\theta \right] \\ &\quad . \left[k_g + \lambda k_g^2 + \lambda\tau_g^2 \right] \left(\frac{ds}{ds_1} \right)^3 \end{aligned} \qquad (28)$$

Then we can give the following corollary.

Corollary 3.1. *Let $\{x, x_1\}$ be Bertrand D-pair. Then the relations between the geodesic curvature k_{g_1} of $x_1(s_1)$ and the geodesic curvature k_g and the geodesic torsion τ_g of $x(s)$ are given as follows:*

i) *If x is a geodesic curve, then the geodesic curvature k_{g_1} of $x_1(s_1)$ is given as follows,*

$$k_{g_1} = \lambda\tau_g^2(\cos\theta - \lambda\tau_g\sin\theta)\left(\frac{ds}{ds_1}\right)^3 . \qquad (29)$$

ii) *If x is a principal line, then the relation between the geodesic curvatures k_{g_1} and k_g is given by*

$$k_{g_1} = k_g(1 + \lambda k_g)^2 \cos\theta \left(\frac{ds}{ds_1}\right)^3 . \qquad (30)$$

Similarly, From the second equation of (3) and by using the fact that g is coincident with g_1, i.e., $n_1 = -\sin\theta T + \cos\theta n$, the geodesic torsion τ_{g_1} of x_1 is given by

$$\begin{aligned} \tau_{g_1} = \Big[&\left(\tau_g + \lambda k_g \tau_g\right)\cos^2\theta \\ &+ \left(k_g + \lambda k_g^2 - \lambda\tau_g^2\right)\sin\theta\cos\theta \\ &- \lambda\tau_g k_g \sin^2\theta \Big] \left(\frac{ds}{ds_1}\right)^2 \end{aligned} \qquad (31)$$

From (31) we can give the following corollary.

Corollary 3.2. *Let $\{x, x_1\}$ be Bertrand D-pair. Then the relations between the geodesic torsion τ_{g_1} of*

$x_1(s_1)$ and the geodesic curvature k_g and the geodesic torsion τ_g of $x(s)$ are given as follows:

***i)** If x is a geodesic curve then the geodesic torsion of x_1 is*

$$\tau_{g_1} = \tau_g \cos\theta\left[\cos\theta - \lambda\tau_g \sin\theta\right]\left(\frac{ds}{ds_1}\right)^2 \quad (32)$$

***ii)** If x is a principal line then the relation between τ_{g_1} and k_g is*

$$\tau_{g_1} = k_g(1+\lambda k_g)\sin\theta\cos\theta\left(\frac{ds}{ds_1}\right)^2 \quad (33)$$

Furthermore, by using (12) and (13), from (32) and (33) we have the following corollary.

***Corollary 3.3. i)** Let $\{x, x_1\}$ be Bertrand D-pair and let x be a geodesic line. Then the geodesic torsion τ_{g_1} of $x_1(s_1)$ is given by*

$$\tau_{g_1} = \tau_g(1-\lambda k_{g_1})\left[(1-\lambda k_{g_1}) + \lambda^2\tau_g\tau_{g_1}\right] \quad (34)$$

***ii)** Let $\{x, x_1\}$ be Bertrand D-pair and let x be a principal line. Then the relation between the geodesic curvatures k_g and k_{g_1} is given as follows*

$$k_g(1+\lambda k_g)(1-\lambda k_{g_1}) = -\frac{1}{\lambda} = constant\,. \quad (35)$$

4. Conclusions

Bertrand partner curves are associated curves with common principal normal vectors and characterized by curvatures and torsions of the curves. In this paper, a different type of associated curves is given by considering associated curves as surface curves and the curves of these new curve pair are called Bertrand partner D-curves. The definition and characterizations of Bertrand partner D-curves are given. Furthermore, the relations between the geodesic curvatures, the normal curvatures and the geodesic torsions of these curves are obtained.

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