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SA Özelliğine Sahip Serbest Modüller Üzerine

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Anahtar kelimeler
Ads özelliği; Dik toplananların kesişim özelliği; Ore özelliği.

Özet

Bir R halkasına, eğer iki dik toplananının arakesiti yine bir dik toplanan ise dik toplananların arakesit özelliğine (SIP) sahiptir denir. Bir M R-modülüne, eğer her $M = A \oplus B$ ayrışımı ve A nın M içindeki her C tümleyeni için $M = A \oplus C$ oluyorsa mutlak dik toplanan özelliğine (ads) sahiptir denir. Bir semisimple sağ Ore bölgesinin kendisi ile dik toplamının, kendi üzerine bir sağ modül olarak, hem SIP hem de ads özelliğini (kısaca, SA özelliğini) sağladığı gösterilmiştir.

On Free Modules with The SA Property

Keywords
Ads property; Summand intersection property; Ore condition.

Abstract

A ring R has the right summand intersection property (SIP) if the intersection of two direct summands of R is also a direct summand. A right R-module M has the absolute direct summand property (ads) if for every decomposition $M = A \oplus B$ of M and every complement C of A in M, we have $M = A \oplus C$. It is shown that the direct sum of two copies of a semisimple right Ore domain has both SIP and ads properties (briefly, SA property) as a right module over itself.

1. Introduction

Throughout the paper all rings are associative with unity and R always denotes such a ring. Modules are unital and for an abelian group M, we use M_R to denote a right R-module. Takıl Mutlu (2015-a) calls an R-module M to have the SA property, if M_R has the SIP and the ads. The motivation for the present study of these properties was provided by the following results:

Kaplansky (1969): a free module over a principal ideal domain, PID, has the SIP,

Quynh and Koşan (2014): A module M is semisimple if and only if every module in $\sigma[M]$ is ads ($\sigma[M]$ denote the Wisbauer category of a module M, i.e. the full category of $R-Mod$ consisting of submodules of quotients of direct summands of copies of M).

These properties have been studied by many authors including Arnold and Hausen (1990), Hausen (1989), Wilson (1986), Burgess and Raphael (1993) and Alahmadi et al. (2012).

We will use $Mat_n(R)$ and R^n to denote the full n-by-n matrix ring over R and the direct sum of n copies of R_R for any positive integer n, respectively. In this paper we consider the following two problems:

(1) If R_R has the SA, find conditions on R to ensure that every free right R-module has the SA.
(2) If R_R has the SA, find conditions on R and κ to ensure that every free right R-module with a basis of cardinality κ has the SA.

It is easy to see that Kaplansky's and Quynh and Koşan's aforementioned results show that both semisimple

Artinian and PID conditions are sufficient conditions for the Problem 1.

2. Results

We observe that from Takıl Mutlu (2015-a, Lemma 2.7) the class of SA right R-modules is closed under direct summands for any ring R. From Takıl Mutlu (2015-b, Theorem 2.6) we recall the following theorem which allows us to use matrix techniques in the study of the aforementioned problems.

Theorem 1 [Takıl Mutlu (2015-b), Theorem 2.6] Let R be any ring with identity, e an idempotent in R such that $R = ReR$ and S the subring eRe and $R = Mat_n(S)$. Then R_R has the SA if and only if S_S^n has the SA.

Since the class of (von Neumann) regular rings is closed with respect to forming full n-by-n matrix rings for any positive integer n, the full n-by-n matrix rings are semisimple. Hence they have the SA. Since every direct summand of an SA-module is SA-module and $R^n \cong e_{11}M_n(R)$ where e_{11} denotes the matrix in R with $(1,1)$ entry 1 and all other entries 0, R^n has the SA. So, Theorem 1 shows that the regularity is a sufficient condition for the Problem 2 when κ is finite. However the following examples show that commutativity, no nonzero zero divisors, ACC on ideals and Krull dimension 2 do not ensure a solution for the Problem 2 when $\kappa \geq 2$.

Example 2 There exists a commutative Noetherian domain R such that $F = R \oplus R$ does not have the SA.
Let $R = \mathbb{Z}$. Then $F = R \oplus R = \mathbb{Z} \oplus \mathbb{Z}$ does not have the ads since $\mathbb{Z}$ is not $\mathbb{Z}$-injective. Hence F does not have the SA.

Example 3 There exists a commutative Noetherian domain R of Krull dimension 2 such that
$F = R \oplus R \oplus R$ does not have the SA.
Let R be as in Smith and Tercan (2004, Example 4), that is
$S = \mathbb{R}[x, y, z]$, $R = S/sS$, where $s = x^2 + y^2 + z^2 - 1$.
Then by Birkenmeier et al. (2006), $F = R \oplus R \oplus R$ does not have the SIP and hence does not have the SA.

Our next result shows that semisimple, the right Ore condition and no nonzero zero divisors do ensure a condition for the Problem 2 when $\kappa = 2$.

Theorem 4 If R is a semisimple right Ore domain, then $(R \oplus R)_R$ has the SA.

Proof. Since R is semisimple, $(R \oplus R)_R$ is injective by Kasch and Wallace (1982, 8.2.2) and hence it is an ads-module. However, $(R \oplus R)_R$ has the SIP by Birkenmeier et al. (2006, Proposition 4). Hence $(R \oplus R)_R$ has the SA.

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