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Approximately Semigroups and Ideals: An Algebraic View of Digital Images

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This article dedicated to honor of July 15 martyrs in Turkey.

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Anahtar kelimeler
Proksimiti uzaylar; Relator uzaylar; Tanımsal yaklaşımlar; Yaklaşımlı yarıgruplar.

Özet

Bu makalede proksimal relator uzaylarında yaklaşımlı yarıgruplar ve ideallere giriş yapılmıştır. Tanımsal proksimiti bağıntısı ile birlikte dikkate alınan dijital görüntülerde yaklaşımlı yarıgrup ve ideal örnekleri verilmiştir. Bundan başka, nesne tanımlaması homomorfizması kullanılarak tanımsal yaklaşımların bazı özellikleri incelenmiştir.

Yaklaşımlı Yarıgruplar ve İdealler: Dijital Görüntülerin Cebirsel İncelenmesi

Keywords
Proximity spaces; Relator spaces; Descriptive approximations; Approximately semigroups.

Abstract

In this article, approximately semigroups and ideals in proximal relator spaces have beenintroduced. In addition to, some examples of approximately semigroups and ideals in digital images endowed with descriptive proximity relation have been given. Furthermore, some properties of descriptively approximations using object descriptive homomorphism have been obtained.

1. Introduction

The concept of ordinary algebraic structures are consist of a nonempty set of abstract points with one or more binary operations which are required to satisfy certain axioms such as a groupoid is an algebraic structure $(A,\circ)$ consist of a nonempty set A and a binary operation "$\circ$" defined on A (Clifford and Preston,1964). And binary operation "$\circ$" must be closed in A whereas in proximal relator spaces, the sets are composed of non-abstract points instead of abstract points and these points are describable with feature vectors in. Descriptively upper approximation of a nonempty set is obtained by using the set of points composed by the proximal relator space together with matching features of points and these are the basic tools for defining algebraic structures on proximal relator spaces and binary operations on any groupoid A in proximal relator space must be closed in descriptively upper approximation of A.

Moreover an example of approximately semigroup on digital images endowed with descriptive proximity relation has given.

2. Preliminaries

Let X be a nonempty set. Family of relations $\mathcal{R}$ on a nonempty set X is called a *relator*. The pair $(X, \mathcal{R})$ (or $X(\mathcal{R})$) is a relator space which is natural generalisations of uniform spaces (Szaz, 1987). If we consider a family of proximity relations on X, we have a proximal relator space $(X, \mathcal{R}_\delta)$ $(X(\mathcal{R}_\delta))$. As in (Peters, 2016), $(\mathcal{R}_\delta)$ contains proximity relations namely, Efremovič proximity δ (Efremovič 1951;1952), Lodato proximity (Lodato, 1962), Wallman proximity, descriptive proximity δ_Φ in defining $\mathcal{R}_{\delta_\Phi}$ (Peters, 2013;Peters et al. 2014).

In this article, we consider the Efremovič proximity δ (Efremovič 1952) and the descriptive proximity δ_Φ in defining a descriptive proximal relator space (denoted by $(X, \mathcal{R}_{\delta_\Phi})$).

An Efremovič proximity δ is a relation on 2^X that satisfies

1. $A\delta B \Rightarrow B\delta A$.

2. $A\delta B \Rightarrow A \neq \emptyset$ and $B \neq \emptyset$.

3. $A \cap B \neq \emptyset \Rightarrow A\delta B$.

4. $A\delta(B \cup C) \Leftrightarrow A\delta B$ or $A\delta C$.

5. $\{x\}\delta\{y\} \Leftrightarrow x = y$.

6. EF axiom. $A\underline{\delta}B \Rightarrow \exists E \subseteq X$ such that $A\underline{\delta}E$ and $E^c\underline{\delta}B$.

Lodato proximity (Lodato, 1962) swaps the EF axiom 2. for the following condition:

$$A\delta B \text{ and } \forall b \in B,$$

$$\{b\}\delta C \Rightarrow A\delta C. \ (Lodato\ Axiom)$$

In a discrete space, a non-abstract point has a location and features that can be measured (Efremovič 1952; Kovăr, 2011). Let X be a nonempty set of non-abstract points in a proximal relator space $(X, \mathcal{R}_{\delta_\Phi})$ and let $\Phi = \{\phi_1, \dots, \phi_n\}$ a set of probe functions where a *probe function* $\Phi: X \to \mathbb{R}$ represents a feature of a sample point in a picture. Let $\Phi(x) = (\phi_1(x), \dots, \phi_n(x))$, $n \in N$ be an object description denote a feature vector for x, which provides a description of each $x \in X$. To obtain a descriptive proximity relation (denoted by δ_Φ), one first choose a set of probe functions.

Definition 2.1.*(**Set Description, Naimpally and Peters, 2013*)*** *Let X be a nonempty set of non-abstract points, Φ an object description and A a subset of X. Then the set description of A is defined as*

$$\mathcal{Q}(A) = \{\Phi(a): a \in A\}.$$

Definition 2.2.*(**Descriptive Set Intersection, Naimpally and Peters, 2013; Peters and Naimpally, 2012*)*** *Let X be a nonempty set of non-abstract points, A and B any two subsets of X. Then the descriptive (set) intersection of A and B is defined as*

$$A \underset{\Phi}{\cap} B = \{x \in A \cup B: \Phi(x) \in \mathcal{Q}(A) \ and \ \Phi(x) \in \mathcal{Q}(B)\}.$$

Definition 2.3.*(Peters, 2013) Let X be a nonempty set of non-abstract points, A and B any two subsets of X. If $\mathcal{Q}(A) \cap \mathcal{Q}(B) \neq \emptyset$, then A is called descriptively near B and denoted by $A\delta_\Phi B$. If $\mathcal{Q}(A) \cap \mathcal{Q}(B) = \emptyset$ then $A\underline{\delta_\Phi} B$ read as A is descriptively far from B.*

Definition 2.4.*(**Descriptive Nearness Collections, Peters, 2013*)*** *Let X be a nonempty set of non-abstract points and A any subset of X. Then the descriptive nearness collection $\xi_\Phi(A)$ is defined by*

$$\xi_\Phi(A) = \{B \in \mathcal{P}(X): A\delta_\Phi B\}.$$

(Peters, et al. 2015) Let $(X, \mathcal{R}_{\delta_\Phi})$ be descriptive proximal relator space and $A \subset X$, where A is consist of non-abstract objects. And let $(A,\cdot)$ and $(\mathcal{Q}(A),\circ)$ be groupoids. Let consider the object description Φ by means of a function

$$\Phi: A \subset X \to \mathcal{Q}(A) \subset \mathbb{R},$$

$$a \mapsto \Phi(a), a \in A.$$

The object description Φ of A in $\mathcal{Q}(A)$ is an *object description homomorphism* if

$$\Phi(a \cdot b) = \Phi(a) \circ \Phi(b) \text{ for all } a, b \in A.$$

Moreover descriptive closure of a point $a \in A$ is defined by

$$cl_\Phi(a) = \{x \in X: \Phi(a) = \Phi(x)\}.$$

Descriptively lower approximation of the set A is consist of $a \in A$ which descrition of descriptive closure $\mathcal{Q}(cl_\Phi(a))$ are subsets of *set description* $\mathcal{Q}(A)$. This discovery process leads to the construction of what is known as the descriptively lower approximation of $A \subseteq X$, which is denoted by Φ_*A.

3. Main Results

Definition 3.1. *(Descriptively Lower Approximation of a Set) Let $(X, \mathcal{R}_{\delta_\Phi})$ be descriptive proximal relator space and $A \subset X$. A descriptively lower approximation of A is defined as*

$$\Phi_* A = \{a \in A: Q(cl_\Phi(a)) \subseteq Q(A)\}.$$

Definition 3.2. *(Descriptively Upper Approximation of a Set) Let $(X, \mathcal{R}_{\delta_\Phi})$ be descriptive proximal relator space and $A \subset X$. A descriptively upper approximation of A is defined as*

$$\Phi^* A = \{x \in X: x\delta_\Phi A\}.$$

Definition 3.3. *(Descriptively Boundary Region) Let $Bnd_\Phi A$ denote the descriptively boundary region of a set $A \subseteq X$ defined by*

$$\Phi_{Bnd} A = \Phi^* A \backslash \Phi_* A = \{x: x \in \Phi^* A \text{ and } x \notin \Phi_* A\}.$$

Lemma 3.4. *Let $(X, \mathcal{R}_{\delta_\Phi})$ be descriptive proximal relator space and $A, B \subset X$, then*

(i) $Q(A \cap B) = Q(A) \cap Q(B)$,

(ii) $Q(A \cup B) = Q(A) \cup Q(B)$.

Proof. (i)

$$Q(A \cap B) = \{\Phi(x): x \in A \cap B\}$$

$$= \{\Phi(x): x \in A\} \cap \{\Phi(x): x \in A\}$$

$$= Q(A) \cap Q(B)$$

(ii)

$$Q(A \cup B) = \{\Phi(x): x \in A \cup B\}$$

$$= \{\Phi(x): x \in A\} \cup \{\Phi(x): x \in A\}$$

$$= Q(A) \cup Q(B)$$

Theorem 3.5. *Let $(X, \mathcal{R}_{\delta_\Phi})$ be descriptive proximal relator space and $A, B \subset X$. Then the following statements hold.*

(1) $(\Phi_* A) \subseteq A \subseteq (\Phi^* A)$,

(2) $\Phi^*(A \cup B) = (\Phi^* A) \cup (\Phi^* B)$,

(3) $\Phi_*(A \cap B) = (\Phi_* A) \cap (\Phi_* B)$,

(4) If $A \subseteq B$, then $(\Phi_* A) \subseteq (\Phi_* B)$,

(5) If $A \subseteq B$, then $(\Phi^* A) \subseteq (\Phi^* B)$,

(6) $\Phi_*(A \cup B) \supseteq (\Phi_* A) \cup (\Phi_* B)$,

(7) $\Phi^*(A \cap B) \subseteq (\Phi^* A) \cap (\Phi^* B)$.

Proof. (1) Let $a \in \Phi_* A$, then $Q(cl_\Phi(a)) \subseteq Q(A)$, where $a \in A$. Hence $(\Phi_* A) \subseteq A$. Let $a \in A$ and it is obvious that $\Phi(a) \in Q(A)$, that is $\Phi(a) \cap Q(A) \neq \emptyset$and so $a\delta_\Phi A$. Therefore $\mathrm{a} \in \Phi^* A$ and then $\mathrm{A} \subseteq \Phi^* A$.

By Lemma 3.4. proofs of (2) and (3) are obvious.

(4) Let $A \subseteq B$, then $A \cap B = A$. From statement (3) we have $\Phi_* A = \Phi_*(A \cap B) = (\Phi_* A) \cap (\Phi_* B)$. Hence $(\Phi_* A) \subseteq (\Phi_* B)$.

(5) Let $A \subseteq B$, then $A \cup B = B$. From statement (2) we get $\Phi^* B = \Phi^*(A \cup B) = (\Phi^* A) \cup (\Phi^* B)$. This implies that $(\Phi^* A) \subseteq (\Phi^* B)$.

(6) Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, by (4) we have $(\Phi_* A) \subseteq \Phi_*(A \cup B)$ and $(\Phi_* B) \subseteq \Phi_*(A \cup B)$. Hence $(\Phi_* A) \cup (\Phi_* B) \subseteq \Phi_*(A \cup B)$.

(7) We know that $A \cap B \subseteq A$ and $A \cap B \subseteq B$. From statement (5) we have $\Phi^*(A \cap B) \subseteq (\Phi^* A)$ and $\Phi^*(A \cap B) \subseteq (\Phi^* B)$. Thus $\Phi^*(A \cap B) \subseteq (\Phi^* A) \cap (\Phi^* B)$.

Definition 3.6. *Let $(X, \mathcal{R}_{\delta_\Phi})$ be descriptive proximal relator space and let "$\cdot$" be binary operation defined on X. A subset G of the set of X is called a descriptive approximately groupoid in descriptive proximal relator space if $x \cdot y \in \Phi^* G$, $\forall x, y \in G$.*

Suppose that G is a *descriptive approximately* groupoid with the binary operation "$\cdot$" in $(X, \mathcal{R}_{\delta_\Phi})$, $g \in G$ and $A, B \subseteq G$. We define the subsets $g \cdot A, A \cdot g, A \cdot B \subseteq \Phi^* G \subseteq X$ as follows:

$$g \cdot A = gA = \{ga: a \in A\},$$

$$A \cdot g = Ag = \{ag: a \in A\},$$

$$A \cdot B = AB = \{ab: a \in A, b \in B\}.$$

Lemma 3.7. *Let (X, δ_Φ) be descriptive proximity space and $A, B \subset X$. If $\Phi: X \to \mathbb{R}$ is an object descriptive homomorphism, then*

$$Q(A)Q(B) = Q(AB).$$

Proof.

$$Q(A)Q(B) = \{\Phi(a)\Phi(b): a \in A, b \in B\} = \{\Phi(ab): a \in A, b \in B\} = Q(AB)$$

Theorem 3.8. *Let (X, δ_Φ) be descriptive proximity space and $A, B \subset X$. If $\Phi: X \to \mathbb{R}$ is an object descriptive homomorphism, then*

$$(\Phi^*A)(\Phi^*B) = \Phi^*(AB).$$

Proof. Suppose that $x \in (\Phi^*A)(\Phi^*B)$, then $x = ab$ with $a \in \Phi^*A$ and $b \in \Phi^*B$. Thus $\Phi(a) \in Q(A)$ and $\Phi(b) \in Q(B)$. Since Φ is an object descriptive homomorphism $\Phi(a)\Phi(b) = \Phi(ab) \in Q(A)Q(B)$ and so by Lemma 3.7. $\Phi(x) \in Q(AB)$, that is $x \in \Phi^*(AB)$. Therefore $(\Phi^*A)(\Phi^*B) \subset \Phi^*(AB)$. Similary we obtain $\Phi^*(AB) \subset (\Phi^*A)(\Phi^*B)$ and so $(\Phi^*A)(\Phi^*B) = \Phi^*(AB)$.

Theorem 3.9 *Let (X, δ_Φ) be descriptive proximity space, $A, B \subset X$. If $\Phi: X \to \mathbb{R}$ is an object descriptive homomorphism, then*

$$(\Phi_*A)(\Phi_*B) \subset \Phi_*(AB).$$

Proof. Let $x \in (\Phi_*A)(\Phi_*B)$, then $x = ab$ with $a \in \Phi_*A$ and $b \in \Phi_*B$. Thus $Q(cl_\Phi(a)) \subseteq Q(A)$, $Q(cl_\Phi(b)) \subseteq Q(B)$ and so $y \in X$ where $\Phi(a) = \Phi(y) \in Q(A)$, $\Phi(b) = \Phi(y) \in Q(B)$. Since Φ is an object descriptive homomorphism $\Phi(a)\Phi(b) = \Phi(ab) = \Phi(y) \in Q(A)Q(B)$ and from Lemma 3.7. $\Phi(ab) = \Phi(y) \in Q(AB)$. Then $Q(cl_\Phi(ab)) \subseteq Q(AB)$, that is $x = ab \in \Phi_*(AB)$. Consequently $(\Phi_*A)(\Phi_*B) \subset \Phi_*(AB)$.

3.1 Approximately Semigroups and Ideals

Definition 3.10. *Let $(X, \mathcal{R}_{\delta_\Phi})$ be descriptive proximal relator space and "$\cdot$" be a binary operation defined on X. $S \subset X$ is called an approximately semigroup in descriptive proximal relator space if the following properties are verified:*

1. $x \cdot y \in \Phi^*S, \forall x, y \in S$.
2. $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ property verify on $\Phi^*S, \forall x, y, z \in S$.

Let an approximately semigroup has approximately identity element $e \in \Phi^*S$ such that $x \cdot e = e \cdot x = x$, $\forall x \in S$. Then S is called an *approximately monoid* in a descriptive proximal relator space.

Let $x \cdot y = y \cdot x, \forall x, y \in S$ property holds in Φ^*S. Then S is *commutative approximately semigroup* in descriptive proximal relator space.

Example 3.11. *Let X be a digital image endowed with descriptive proximity relation δ_Φ and consists of 25 pixels as in Fig. 1.*

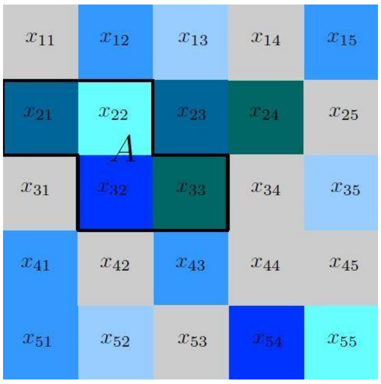

Figure 1: Digital image

A pixel x_{ij} is an element at position (i, j) (row and column) in digital image X. Let ϕ be a probe function that represent RGB colour of each pixel are given in Table 1.

Table 1. RGB codes of each pixel in Fig. 1.

	Red	Green	Blue		Red	Green	Blue
x_{11}	204	204	204	x_{34}	204	204	204
x_{12}	51	153	255	x_{35}	204	255	255
x_{13}	204	255	255	x_{41}	51	153	255
x_{14}	204	204	204	x_{42}	204	204	204
x_{15}	51	153	255	x_{43}	51	153	255
x_{21}	0	102	153	x_{44}	204	204	204
x_{22}	102	255	255	x_{45}	204	204	204

x_{23}	0	102	153	x_{51}	51	153	255
x_{24}	0	102	102	x_{52}	204	255	255
x_{25}	204	204	204	x_{53}	204	204	204
x_{31}	204	204	204	x_{54}	0	51	255
x_{32}	0	51	255	x_{55}	102	255	255
x_{33}	0	102	102				

Let

$$\cdot : X \times X \to X$$

$$(x_{ij}, x_{kl}) \mapsto x_{ij} \cdot x_{kl} = x_{pr}$$

where $p = \min\{i, k\}$ and $r = \min\{j, l\}$

be a binary operation on X and $A = \{x_{21}, x_{22}, x_{32}, x_{33}\}$be a subimage (subset) of X.

We can compute the descriptively upper approximation of A by using the Definition 3.2. $\Phi^*A = \{x_{ij} \in X: \delta_\phi A\}$, where $Q(A) = \{\phi(x_{ij}): x_{ij} \in A\}$. Then $\phi(x_{ij}) \cap Q(A) \neq \emptyset$ such that $x_{ij} \in X$. From Table 1, we obtain

$$Q(A) = \{\phi(x_{21}), \phi(x_{22}), \phi(x_{32}), \phi(x_{33})\}$$
$$= \{(0,102,153), (102,255,255), (0,51,255), (0,102,102)\}$$

Hence we get

$$\Phi^*A = \{x_{21}, x_{22}, x_{23}, x_{24}, x_{32}, x_{33}, x_{54}, x_{55}\}$$

as shown in Fig. 2.

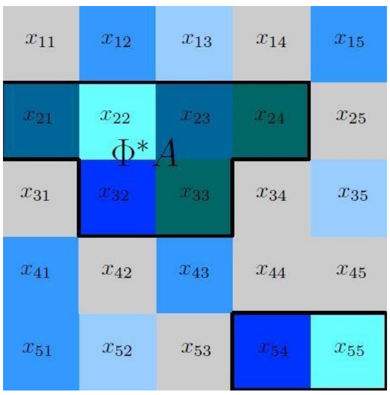

Figure 2: Φ^*A

Since

1. For all $x_{ij}, x_{kl} \in A$, $x_{ij} \cdot x_{kl} \in \Phi^*A$,

2. For all $x_{ij}, x_{kl}, x_{mn} \in A$, $(x_{ij} \cdot x_{kl}) \cdot x_{mn} = x_{ij} \cdot (x_{kl} \cdot x_{mn})$ property holds in Φ^*A,

are satisfied, the subimage A of the digital image X is indeed an approximately semigroup in descriptive proximity space (X, δ_Φ) with binary operation " $\cdot$ ". Moreover, since $x_{ij} \cdot x_{kl} = x_{kl} \cdot x_{ij}$, for all $x_{ij}, x_{kl} \in A$ property holds in Φ^*A, A is a commutative approximately semigroup.

Notice that in Example 3.11 proximal identity element is not unique.x_{33} and $x_{55} \in \Phi^*A$ have feature ofa *proximal identity element*. So Adoes nothave an unique identity element and$A$ is not a commutative approximately monoid.

Example 3.12. *Let X be a digital image endowed with descriptive proximity relation δ_Φ and consists of* 16 *pixels as in Fig. 3.*

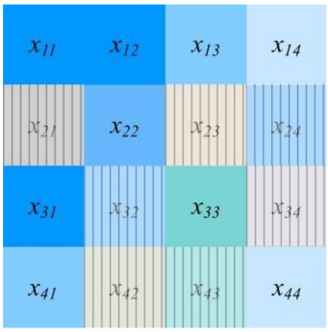

Figure 3: Digital image X and $S \subset X$

A pixel x_{ij} is an element at position (i, j) (row and column) in X. Let ϕ be a probe function that represents RGB colour of each pixel are given in Table 2.

Table 2. RGB codes of each pixel in Fig. 3.

	Red	Green	Blue		Red	Green	Blue
x_{11}	0	151	255	x_{31}	0	151	255
x_{12}	0	151	255	x_{32}	103	183	255
x_{13}	130	255	255	x_{33}	121	212	211

x_{14}	200	204	255	x_{34}	205	205	216
x_{21}	170	153	170	x_{41}	130	205	255
x_{22}	103	102	255	x_{42}	202	210	187
x_{23}	224	255	187	x_{43}	121	212	211
x_{24}	103	102	255	x_{44}	200	230	255

Let

$$\cdot : X \times X \to X$$

$$(x_{ij}, x_{kl}) \mapsto x_{ij} \cdot x_{kl} = x_{pr},$$

$$p = \min\{i, k\} \text{ and } r = \min\{j, l\}$$

be a binary operation on X and

$$S = \{x_{21}, x_{23}, x_{24}, x_{32}, x_{34}, x_{42}, x_{43}\}$$

a subimage (subset) of X.

We can compute the descriptively upper approximation of S by using the Definition 3.2. $\Phi^* S = \{x_{ij} \in X : x_{ij} \delta_\phi S\}$, where $Q(S) = \{\phi(x_{ij}) : x_{ij} \in S\}$. Then $\phi(x_{ij}) \cap Q(S) \neq \emptyset$ such that $x_{ij} \in X$. From Table 2, we obtain:

$$\begin{aligned} Q(S) &= \{\phi(x_{21}), \phi(x_{23}), \phi(x_{24}), \phi(x_{32}), \\ &\quad \phi(x_{34}), , \phi(x_{42}), \phi(x_{43})\} \\ &= \{(170,170,170), (224,208,187), \\ &\quad (103,183,255), (205,205,216), \\ &\quad (202,210,187), (121,212,211)\} \end{aligned}$$

Hence we get

$$\Phi^* S = \{x_{21}, x_{22}, x_{23}, x_{24}, x_{32}, x_{33}, x_{34}, x_{42}, x_{43}\}$$

as shown in Fig. 4.

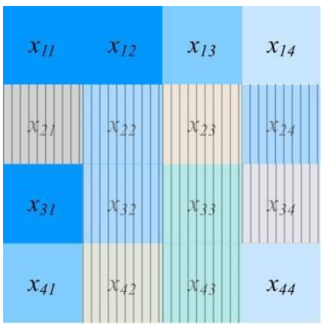

Figure 4: $\Phi^* S$

by Definition 3.10., since

1. For all $x_{ij}, x_{kl} \in S$, $x_{ij} \cdot x_{kl} \in \Phi^* S$,

2. For all $x_{ij}, x_{kl}, x_{mn} \in A$, $(x_{ij} \cdot x_{kl}) \cdot x_{mn} = x_{ij} \cdot (x_{kl} \cdot x_{mn})$ property holds in $\Phi^* S$,

are satisfied, the subimage S of the digital image X is indeed an approximately semigroup in descriptive proximity space (X, δ_Φ) with binary operation " $\cdot$ ". Moreover, since $x_{ij} \cdot x_{kl} = x_{kl} \cdot x_{ij}$, for all $x_{ij}, x_{kl} \in S$ property holds in $\Phi^* S$, S is a commutative approximately semigroup.

Definition 3.13. *Let T be a nonempty subset of approximately semigroup S in $(X, \mathcal{R}_{\delta_\Phi})$. T is called an approximately subsemigroup of S if $TT \subseteq \Phi^* T$. In other words, T is an approximately semigroup with the binary operation of S restricted to T.*

Example 3.14. *From Example 3.12., let we consider approximately semigroup $S = \{x_{21}, x_{23}, x_{24}, x_{32}, x_{34}, x_{42}, x_{43}\}$ in descriptive proximity space (X, δ_Φ) with binary operation " $\cdot$ ".*

Let $T = \{x_{21}, x_{23}, x_{32}\}$ be a subimage (subset) of $S \subset X$. We can compute the descriptively upper approximation of T by using the Definition 3.2. $\Phi^* T = \{x_{ij} \in X : x_{ij} \delta_\phi T\}$, where $Q(T) = \{\phi(x_{ij}) : x_{ij} \in T\}$. Then $\phi(x_{ij}) \cap Q(T) \neq \emptyset$ such that $x_{ij} \in X$. By Table 2, we obtain

$$\begin{aligned} Q(T) &= \{\phi(x_{21}), \phi(x_{23}), \phi(x_{32})\} \\ &= \{(170,170,170), (224,208,187), \\ &\quad (103,183,255)\}. \end{aligned}$$

Then we get $\Phi^* T = \{x_{21}, x_{22}, x_{23}, x_{32}\}$. By Definition 3.13., since $TT \subseteq \Phi^* T$ property holds, the subimage T of the digital image $S \subset X$ is indeed an approximately subsemigroup in descriptive proximity space (X, δ_Φ) with binary operation " $\cdot$ ".

Definition 3.15. *Let $(X, \mathcal{R}_{\delta_\Phi})$ be descriptive proximal relator space, S be a approximately semigroup and $\emptyset \neq I \subseteq S$.*

(1) I is called an approximately left ideal of S if Φ^*I be a left ideal of S, that is $S(\Phi^*I) \subseteq \Phi^*I$.

(2) I is called an approximately right ideal of S if Φ^*I is a right ideal of S, that is $(\Phi^*I)S \subseteq \Phi^*I$.

(3) I is called an approximately bi-ideal of S if Φ^*I is a bi-ideal of S, that is $(\Phi^*I)S(\Phi^*I) \subseteq \Phi^*I$.

Example 3.16. *From Example 3.12., let we consider approximately semigroup $S = \{x_{21}, x_{23}, x_{24}, x_{32}, x_{34}, x_{42}, x_{43}\}$ in descriptive proximity space (X, δ_Φ) with binary operation " $\cdot$ ".*

Let $I = \{x_{21}, x_{23}, x_{24}, x_{32}, x_{42}\}$ be a subimage (subset) of $S \subset X$. We can compute the descriptively upper approximation of I by Definition 3.2. $\Phi^*I = \{x_{ij} \in X: x_{ij}\delta_\phi I\}$, where $Q(I) = \{\phi(x_{ij}): x_{ij} \in I\}$. Then $\phi(x_{ij}) \cap Q(I) \neq \emptyset$ such that $x_{ij} \in X$. From Table 2, we obtain

$$\begin{aligned} Q(I) &= \{\phi(x_{21}), \phi(x_{23}), \phi(x_{24}), \phi(x_{32}), \phi(x_{42})\} \\ &= \{(170,170,170), (224,208,187), \\ &\quad (103,183,255), (202,210,187)\} \end{aligned}$$

Then we get $\Phi^*I = \{x_{21}, x_{22}, x_{23}, x_{24}, x_{32}, x_{42}\}$. By Definition 3.15., since $S(\Phi^*I) \subseteq \Phi^*I$ property holds, the subimage I is indeed an approximately left ideal of the digital image S in descriptive proximity space (X, δ_Φ) with binary operation " $\cdot$ ". Furthermore, since S is a commutative approximately semigroup, we observe that I is also approximately right and bi-ideal of S.

Example 3.17. *By Examples 3.12. and 3.14., let we consider approximately subsemigroup $T = \{x_{21}, x_{23}, x_{32}\} \subset S$ in descriptive proximity space (X, δ_Φ) with binary operation " $\cdot$ ".*

By Example 3.14. we know that descriptively upper approximation of T is $\Phi^*T = \{x_{21}, x_{22}, x_{23}, x_{32}\}$. Then by Definition 3.15., since $S(\Phi^*T) \subseteq \Phi^*T$ property holds, the approximately subsemigroup (subimage) T is indeed an approximately left ideal of the digital image S in descriptive proximity space (X, δ_Φ) with binary operation " $\cdot$ ". Furthermore, since S is a commutative approximately semigroup, we observe that T is also approximately right and bi-ideal of S.

In Example 3.17., we observe that the approximately subsemigroup (subimage) T is indeed an approximately left ideal of S in (X, δ_Φ).

Example 3.18. *From Example 3.12., let we consider approximately semigroup $S = \{x_{21}, x_{23}, x_{24}, x_{32}, x_{34}, x_{42}, x_{43}\}$ in descriptive proximity space (X, δ_Φ) with binary operation " $\cdot$ ".*

Let $K = \{x_{34}, x_{43}\}$ be a subimage (subset) of $S \subset X$. We can compute the descriptively upper approximation of K by using the Definition 3.2. $\Phi^*K = \{x_{ij} \in X: x_{ij}\delta_\phi K\}$, where $Q(K) = \{\phi(x_{ij}): x_{ij} \in K\}$. Then $\phi(x_{ij}) \cap Q(K) \neq \emptyset$ such that $x_{ij} \in X$. From Table 2, we obtain

$$\begin{aligned} Q(K) &= \{\phi(x_{34}), \phi(x_{43})\} \\ &= \{(205,205,216), (121,212,211)\} \end{aligned}$$

Then we get $\Phi^*K = \{x_{34}, x_{33}, x_{43}\}$. By Definition 3.13., since $KK \subseteq \Phi^*K$ property holds, the subimage K of the digital image $S \subset X$ is indeed an approximately subsemigroup in (X, δ_Φ) with binary operation " $\cdot$ ". But since

$$S(\Phi^*K)\{x_{21}, x_{22}, x_{23}, x_{24}, x_{32}, x_{33}, x_{34}, x_{42}, x_{43}\} \square \Phi^*K$$

subimage K is not an approximately left ideal (right or bi-ideal) of S in (X, δ_Φ).

In addition, although a subimage (subset) $J = \{x_{24}, x_{42}\}$ of $S \subset X$ is also an approximately subsemigroup of S, it is not an approximately left ideal (right or bi-ideal) of S in (X, δ_Φ).

Theorem 3.19 *Let $(X, \mathcal{R}_{\delta_\Phi})$ be descriptive proximal relator space and $S \subseteq X$.*

(1) If S is a semigroup in X, then S is an *approximately semigroup* in descriptive proximal relator space.

(2) If I is a left (right, bi) ideal of *approximately semigroup* S, then I is an *approximately* left (right, bi) ideal of S.

Proof. (1) Suppose that $S \subseteq X$ be a semigroup. From Theorem 3.5.(1), $\emptyset \neq S \subseteq \Phi^*S$. Hence $x \cdot y \in \Phi^*S$, $\forall x, y \in S$ and $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ property holds in Φ^*S, $\forall x, y, z \in S$. Then S is an *approximately semigroup* in descriptive proximal relator space.

(2) Suppose that I be a left ideal of *approximately semigroup* S, that is $SI \subseteq I$. We know that $S \subseteq \Phi^*S$. Hence, by Theorems 3.5.(5) and 3.8.,

$$S(\Phi^*I) \subseteq (\Phi^*S)(\Phi^*I) = \Phi^*(SI) \subseteq \Phi^*I.$$

As a result Φ^*I is a left ideal of *approximately semigroupS*, and so I is an *approximately* left ideal of S. We can easily prove that I is a *approximately* right ideal of S. Therefore, I is an *approximately* left, right or bi-ideal of S.

The Theorem 3.19. prove that the concept of *approximately semigroup* (left, right or bi-ideal) is a generalized concept of a semigroup (left, right or bi-ideal).

Theorem 3.20. *Let* $(X, \mathcal{R}_{\delta_\Phi})$ *be descriptive proximal relator space and* $S \subseteq X$ *be a semigroup and* $A \subseteq S$.

(1) If A is a subsemigroup of S, then $\emptyset \neq \Phi_*A$ is a subsemigroup of S.

(2) If I is a left (right, bi) ideal of S, then $\emptyset \neq \Phi_*I$ is a left (right, bi) ideal of Φ_*S.

Proof. (1) Suppose that A be a subsemigroup of S, then by Theorems 3.9. and 3.5.(4),

$$(\Phi_*A)(\Phi_*A) \subseteq \Phi_*(AA) \subseteq \Phi_*A.$$

Consequently $\emptyset \neq \Phi_*A$ is a subsemigroup of $S \subseteq X$.

(2) Suppose that I is a left ideal of S, that is $SI \subseteq I$. Then, by Theorems 3.9. and 3.5.(4),

$$(\Phi_*S)(\Phi_*I) \subseteq \Phi_*(SI) \subseteq \Phi_*I.$$

As a result $\emptyset \neq \Phi_*I$ is a left ideal of Φ_*S. The proofs of other cases can be written in a similar processes.

Theorem 3.21. *Let* $(X, \mathcal{R}_{\delta_\Phi})$ *be descriptive proximal relator space and* $S \subseteq X$. *If* I *is a bi-ideal of* S, *then it is an approximately bi-ideal of* S.

Proof. Suppose that I is a bi-ideal of S. Then, by Theorems 3.8. and 3.5.(5),

$$(\Phi^*I)(S)(\Phi^*I) \subseteq (\Phi^*I)(\Phi^*S)(\Phi^*I) \subseteq \Phi^*(ISI) \subseteq \Phi^*I.$$

As a result, by Theorem 3.20.(2), Φ^*I is a bi-ideal of S, that is, I is an *approximately* bi-ideal of S.

Theorem 3.22. *Let* $(X,\mathcal{R}_{\delta_\Phi})$ *be descriptive proximal relator space and* $S \subseteq X$. *If* I *is a bi-ideal of* S, *then* $\emptyset \neq \Phi_*I$ *is a bi-ideal of* Φ_*S.

Proof. Suppose that I is a bi-ideal of S. Then, by Theorems 3.9 and 3.5.(6),

$$(\Phi_*I)(\Phi_*S)(\Phi_*I) \subseteq \Phi_*(ISI) \subseteq \Phi_*I.$$

Consequently, by Theorem 3.20.(2), $\emptyset \neq \Phi_*I$ is a bi-ideal of Φ_*S.

Theorem 3.23. *Let* $(X, \mathcal{R}_{\delta_\Phi})$ *be descriptive proximal relator space and* $S \subseteq X$. *If* I *and* J *are right and left ideals of* S, *respectively, then*

$$\Phi^*(IJ) \subseteq (\Phi^*I) \cap (\Phi^*J).$$

Proof. Suppose that I and J are right and left ideals of S, respectively. Then $IJ \subseteq IS \subseteq I$ and $IJ \subseteq SJ \subseteq J$. Thus $IJ \subseteq I \cap J$. As a result, by Theorem 3.5.(5) and (7),

$$\Phi^*(IJ) \subseteq \Phi^*(I \cap J) \subseteq (\Phi^*I) \cap (\Phi^*J).$$

Theorem 3.24. *Let* $(X, \mathcal{R}_{\delta_\Phi})$ *be descriptive proximal relator space and* $S \subseteq X$. *If* I *and* J *are right and left ideals of* S, *respectively, then*

$$\Phi_*(IJ) \subseteq (\Phi_*I) \cap (\Phi_*J).$$

Proof. Suppose that I and J are right and left ideals of S, respectively. Then $IJ \subseteq IS \subseteq I$ and $IJ \subseteq SJ \subseteq J$. Thus $IJ \subseteq I \cap J$. Consequently, by Theorem 3.5.(3) and (4),

$$\Phi_*(IJ) \subseteq \Phi_*(I \cap J) \subseteq (\Phi_*I) \cap (\Phi_*J).$$

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