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Kesirli Fark Operatörü ile Elde Edilen Dizilerin Bazı Cesàro-Tipi Toplanabilirliği ve İstatistiksel Yakınsaklığı

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Anahtar kelimeler
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Özet

Bu çalışmada, reel (ya da kompleks) değerli dizilerin kuvvetli (p,Δ^{α})-Cesàro toplanabilmesi ve Δ^{α}-istatistiksel yakınsaklığı verilmiştir. Δ^{α}-istatistiksel yakınsaklık ve kuvvetli (p,Δ^{α})-Cesàro toplanabilme arasındaki bazı kapsama ilişkiler incelenmiştir. Ayrıca $w_p(\Delta^{\alpha}, f)$ ve $S(\Delta^{\alpha})$ uzayları arasındaki bazı kapsama bağıntıları verilmiştir.

Some Cesàro-Type Summability and Statistical Convergence of Sequences Generated by Fractional Difference Operator

Keywords

Fractional Difference operator ; Statistical Convergence ; Cesàro Summability.

Abstract

In this paper, strong (p,Δ^{α})-Cesàro summability and Δ^{α}-statistical convergence are introduced for real (or complex) valued sequences. Some inclusion relations between the Δ^{α}-statistical convergence and strong (p,Δ^{α})-Cesàro summability are examined. Further inclusion relations between the spaces $w_p(\Delta^{\alpha}, f)$ and $S(\Delta^{\alpha})$ are introduced.

1. Preliminaries and Background

Zygmund (1979) gave the idea of statistical convergence. Also Steinhaus (1951) and Fast (1951) introduced the statistical convergence. Later it was reintroduced by Schoenberg (1959) independently. This concept has been applied in the theory of Fourier analysis, interval analysis, trigonometric series, number theory, measure theory, ergodic theory, Fuzzy set theory and Banach spaces. Fridy (1985), Connor (1988), Salat (1980), Edely (2009) and many others linked this notion with summability theory. We denote the space of all real (or complex) valued sequences by w. Any subspace of w is called a sequence space. The set of all linear spaces of null, convergent and bounded sequences $x=(x_k)$ (real or complex terms) denoted by c_0, c and l_∞, respectively. These are normed spaces by $\|x\|_\infty = \sup_k |x_k|$ and also Banach spaces, while k belongs to natural numbers which is the set of $\{1,2,3,...\}$.

The notion of statistical convergence is given depending on the density of subsets of natural numbers. The definition of it for B which is subset of natural numbers is given by

$$\delta(B)=\lim_{n\to\infty}\frac{1}{n}\left|\{k\le n \ : \ k\in B\}\right|.$$

Here $\left|\{k \le n \,:\, k \in B\}\right|$ indicates the number of elements of B not exceeding n. B has 0 natural density and $\delta(B^c) = 1 - \delta(B)$, for B is any finite subset. A sequence (x_k) is called statistically convergent to l if for every $\varepsilon > 0$,

$$\delta(\{k : |x_k - \mathrm{l}| \ge \varepsilon\}) = 0 .$$

It will be written by $S - \lim x_k = \mathrm{l}$. S indicate the set of sequences which are statistically convergent.

A sequence (x_k) is called strongly Cesàro summable to l if

$$\lim_{n\to\infty} \tfrac{1}{n} \sum_{k=1}^{n} |x_k - \mathrm{l}| = 0 .$$

The set of sequences which are strongly Cesàro summable is indicated by $[C,1]$. This set is given as

$$[C,1] = \left\{ x = (x_k) \,:\, \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |x_k - \mathrm{l}| = 0, \text{for some } \mathrm{l} \right\}.$$

Kızmaz (1981) introduced difference sequence spaces

$$X(\Delta) = \{x = (x_k) \in w \,:\, \Delta x \in X\}$$

for $X = \mathrm{l}_\infty, c, c_0$ where $\Delta x = (x_k - x_{k+1})$. Then, generalized concept of this idea was given by Et and Çolak (1995) as below:

$$X(\Delta^m) = \{x = (x_k) \in w \,:\, \Delta^m x \in X\}$$

where $\Delta^m = \sum_{i=0}^{m} (-1)^i \begin{pmatrix} m \\ i \end{pmatrix} x_{k+i}$, m belongs to natural numbers and $X = \mathrm{l}_\infty, c$ and c_0. Also Bektaş et al. (2004), Bektaş and Et (2006), Et (2000), Et and Nuray (2001), Işık (2004) and other authors generalized this set.

Let $\Gamma(\alpha)$ denote the Euler Gamma function of a real number α and $\alpha \notin \{0, -1, -2, -3, \ldots\}$. By the definition, it can be denoted as an improper integral as follows:

$$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha - 1} dt.$$

For a proper fraction α, Baliarsingh (2013), Baliarsingh and Dutta (2015) defined the generalized fractional difference operator $\Delta^\alpha \,:\, w \to w$ as follows:

$$\Delta^\alpha x_k = \sum_{i=0}^{\infty} (-1)^i \frac{\Gamma(\alpha+1)}{i!\,\Gamma(\alpha - i + 1)} x_{k+i}. \qquad (1)$$

This concept is a generalization of Δ^m operator. We assume throughout the paper that (1) is a convergent serie. If alpha is positive integer, (1) reduce to finite sum, that is,

$$\sum_{i=0}^{\alpha} (-1)^i \frac{\Gamma(\alpha+1)}{i!\,\Gamma(\alpha - i + 1)} x_{k+i}.$$

For instance,

$$\Delta^{1/2} x_k = x_k - \frac{1}{2} x_{k+1} - \frac{1}{8} x_{k+2} - \frac{1}{16} x_{k+3} - \frac{5}{128} x_{k+4} - \frac{7}{256} x_{k+5} \cdots ,$$
$$\Delta^{2/3} x_k = x_k - \frac{2}{3} x_{k+1} - \frac{1}{9} x_{k+2} - \frac{4}{81} x_{k+3} - \frac{7}{243} x_{k+4} - \frac{14}{729} x_{k+5} \cdots .$$

Baliarsingh and Dutta (2015) gave the fractional order difference sequence spaces as follows:

$$\mathrm{l}_\infty(\Gamma, \Delta^\alpha, p) = \{x = (x_k) \in w \,:\, \sup_k |\Delta^\alpha x_k|^{p_k} < \infty\},$$
$$c_0(\Gamma, \Delta^\alpha, p) = \{x = (x_k) \in w \,:\, \lim_{k\to\infty} |\Delta^\alpha x_k|^{p_k} = 0\},$$
$$c(\Gamma, \Delta^\alpha, p) = \{x = (x_k) \in w \,:\, \lim_{k\to\infty} |\Delta^\alpha x_k - \mathrm{l}|^{p_k} = 0 \text{ for some } \mathrm{l} \in \mathrm{C}\}$$

where α is a proper fraction, Δ^α is defined by (1) and $p = (p_k)$ is a bounded sequence of positive real numbers.

2. Main Results

In this part we introduce the notion of Δ^α-statistical convergence and the strong (p, Δ^α)-Cesàro summability. Also introduce some relations between these notions and we examine some inclusion relations between new spaces.

Definition 2.1 The sequence (x_k) is said to be Δ^α-statistically convergent if there is a complex number l such that

$$\lim_{n\to\infty}\frac{1}{n}\left|\left\{k\le n\ :\ \left|\Delta^\alpha x_k - l\right|\ge\varepsilon\right\}\right| = 0.$$

In this case, x is Δ^α-statistically convergent to l. It will be written by $S(\Delta^\alpha)-\lim x_k = l$. The set of sequences which are Δ^α-statistically convergent is denoted by $S(\Delta^\alpha)$.

Theorem 2.2 $x=(x_k)$, $y=(y_k)$ are sequences of real or complex numbers.

(i) If $S(\Delta^\alpha)-\lim x_k = x_0$ and c belongs to complex numbers, then $S(\Delta^\alpha)-\lim cx_k = cx_o$.

(ii) If $S(\Delta^\alpha)-\lim x_k = x_0$, $S(\Delta^\alpha)-\lim y_k = y_0$, then $S(\Delta^\alpha)-\lim(x_k+y_k) = x_0+y_0$.

Proof. (i) In case $c=0$, it is seen eaisly. Let assume $c\ne 0$, then we have desired result from

$$\frac{1}{n}\left|\left\{k\le n\ :\ \left|\Delta^\alpha cx_k - cx_0\right|\ge\varepsilon\right\}\right|\le\frac{1}{n}\left|\left\{k\le n\ :\ \left|\Delta^\alpha x_k - x_0\right|\ge\frac{\varepsilon}{|c|}\right\}\right|.$$

(ii) It is seen from following inequality;

$$\frac{1}{n}\left|\left\{k\le n\ :\ \left|\Delta^\alpha(x_k+y_k)-(x_0+y_0)\right|\ge\varepsilon\right\}\right|\le\frac{1}{n}\left|\left\{k\le n\ :\ \left|\Delta^\alpha x_k - x_0\right|\ge\frac{\varepsilon}{2}\right\}\right| + \frac{1}{n}\left|\left\{k\le n\ :\ \left|\Delta^\alpha y_k - y_0\right|\ge\frac{\varepsilon}{2}\right\}\right|.$$

Theorem 2.3 The inclusion $c(\Gamma,\Delta^\alpha,p)\subset S(\Delta^\alpha)$ holds and the inclusion is strict, where $p_k=1$.

Proof. Since $c\subset S$, then $c(\Gamma,\Delta^\alpha,p)\subset S(\Delta^\alpha)$. To prove strictness let choose $x=(x_k)$ by

$$\Delta^\alpha x_k = \begin{cases} 1, & k=n^3 \\ -1, & k+1=n^3 \\ 0, & otherwise \end{cases} \tag{2}$$

Then, we have $x\in S(\Delta^\alpha)$ but $x\notin c(\Gamma,\Delta^\alpha,p)$.

Theorem 2.4 If $x=(x_k)$ is Δ^α-statistically convergent sequence, then, it is a Δ^α-statistically Cauchy sequence.

Proof. Let assume x is Δ^α-statistical convergent to l and $\varepsilon>0$. Then, we have $\left|\Delta^\alpha x_k - l\right|<\varepsilon/2$ for almost all k. Let choose n, then,

$$\left|\Delta^\alpha x_n - l\right|<\varepsilon/2$$

holds. Hence, we have

$$\begin{aligned}\left|\Delta^\alpha x_k - \Delta^\alpha x_n\right| &< \left|\Delta^\alpha x_k - l\right| + \left|\Delta^\alpha x_n - l\right| \\ &< \varepsilon\end{aligned}$$

for almost all k. Therefore, x is Δ^α-statistical Cauchy sequence.

Theorem 2.5 Although $S(\Delta^\alpha)$ and $l_\infty(\Gamma,\Delta^\alpha,p)$ overlap, neither of $S(\Delta^\alpha)$ and $l_\infty(\Gamma,\Delta^\alpha,p)$ includes the other where $p_k=1$.

Proof. If we choose $x=(x_k)$ as given by (2), then, $x\in S(\Delta^\alpha)$ but $x\notin l_\infty(\Gamma,\Delta^\alpha,p)$. Now we choose $x=(1,0,1,0,1,\ldots)$ then $\Delta^\alpha x_k = (-1)^k 2^{\alpha-1}$ and $x\in l_\infty(\Gamma,\Delta^\alpha,p)$ but $x\notin S(\Delta^\alpha)$.

Theorem 2.6 Although $S(\Delta^\alpha)$ and l_∞ overlap, neither of $S(\Delta^\alpha)$ and l_∞ includes the other.

Proof. One can see following similar way given in proof of Theorem 2.5.

Theorem 2.7 $S\cap S(\Delta^\alpha)\ne\varnothing$ holds.

Proof. Let choose $x=(1,1,1,\ldots)$. Since $x\in S$ and $\Delta^\alpha x_k = 0$, $x\in S(\Delta^\alpha)$ the intersection is non-empty.

Definition 2.8 Let p be a positive real number. A sequence $x=(x_k)$ is said to be strongly (p,Δ^α)-Cesàro summable, if there is a real (or complex) number l such that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left|\Delta^{\alpha}x_k-1\right|^{p}=0.$$

In this case, x is strongly (p,Δ^{α})-Cesàro summable, to 1. The set of sequences which are all strongly (p,Δ^{α})-Cesàro summable is denoted by $w_p(\Delta^{\alpha})$.

Theorem 2.9 The inclusion $w_q(\Delta^{\alpha})\subset w_p(\Delta^{\alpha})$ holds for $0<p<q<\infty$.

Proof. It follows from Hölder's inequality.

Theorem 2.10 If $x=(x_k)$ is strongly (p,Δ^{α})-Cesàro summable to 1, then it is Δ^{α}-statistically convergent to 1, where $0<p<\infty$.

Proof. For any sequence $x=(x_k)$ and $\varepsilon>0$, we have

$$\begin{aligned}\sum_{k=1}^{n}\left|\Delta^{\alpha}x_k-1\right|^{p} &= \sum_{\substack{k=1\\ |x_k-L|\geq\varepsilon}}^{n}\left|\Delta^{\alpha}x_k-1\right|^{p}+\sum_{\substack{k=1\\ |x_k-L|<\varepsilon}}^{n}\left|\Delta^{\alpha}x_k-1\right|^{p}\\ &\geq\sum_{k=1}^{n}\left|\Delta^{\alpha}x_k-1\right|^{p}\geq\left|\left\{k\leq n\ :\ \left|\Delta^{\alpha}x_k-1\right|\geq\varepsilon\right\}\right|.\varepsilon^{p}\end{aligned}$$

and so

$$\frac{1}{n}\sum_{k=1}^{n}\left|\Delta^{\alpha}x_k-1\right|^{p}\geq\frac{1}{n}\left|\left\{k\leq n\ :\ \left|\Delta^{\alpha}x_k-1\right|\geq\varepsilon\right\}\right|.\varepsilon^{p}.$$

From this if $x=(x_k)$ is (p,Δ^{α})-Cesàro summable to 1, then, it is Δ^{α}-statistically convergent to 1.

Corollary 2.11 If $x=(x_k)$, which is Δ^{α}-bounded sequence, is Δ^{α}-statistically convergent to 1, also it is strongly (p,Δ^{α})-Cesàro summable to 1.

3. Statistical convergence and new sequence space defined by using Modulus function

Nakano (1953) introduced the notion of modulus. It is defined $f\ :[0,\infty)\to[0,\infty)$ and it has properties as follow:

(i) $f(x)=0\Leftrightarrow x=0$,

(ii) $f(x+y)\leq f(x)+f(y)$ while $x,y\geq 0$,

(iii) f is a continuous function from the right at 0.

(iv) f is increasing function.

Definition 3.1 Let $p=(p_k)$ be a sequence of strictly positive real numbers and f be a modulus function. Now we give the definition of following sequence space:

$$w_p(\Delta^{\alpha},f)=\left\{x=(x_k)\in w\ :\ \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left[f\left(\left|\Delta^{\alpha}x_k-1\right|\right)\right]^{p_k}=0\right\}.$$

We assume $p=(p_k)$ is bounded and $0<h=\inf_k p_k\leq\sup_k=H<\infty$.

Theorem 3.2 The inclusion $w_p(\Delta^{\alpha},f)\subset S(\Delta^{\alpha})$ is strictly holds for any modulus function f.

Proof. Let $x\in w_p(\Delta^{\alpha},f)$, $\varepsilon>0$ and f be a modulus function. $\sum_1$ and $\sum_2$ denote the sums over $k\leq n$ with

$$\left|\Delta^{\alpha}x_k-1\right|\geq\varepsilon \text{ and } \left|\Delta^{\alpha}x_k-1\right|<\varepsilon,$$

respectively. Then,

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n}\left[f\left(\left|\Delta^{\alpha}x_k-1\right|\right)\right]^{p_k} &=\frac{1}{n}\sum_{1}\left[f\left(\left|\Delta^{\alpha}x_k-1\right|\right)\right]^{p_k}\\ &\geq\frac{1}{n}\sum_{1}\left[f(\varepsilon)\right]^{p_k}\\ &\geq\frac{1}{n}\sum_{1}\min\left(\left[f(\varepsilon)\right]^{h},\left[f(\varepsilon)\right]^{H}\right)\\ &\geq\frac{1}{n}\left|\left\{k\leq n\ :\ \left|\Delta^{\alpha}x_k-1\right|\geq\varepsilon\right\}\right|\min\left(\left[f(\varepsilon)\right]^{h},\left[f(\varepsilon)\right]^{H}\right).\end{aligned}$$

Hence, $x\in S(\Delta^{\alpha})$. To prove strictly let choose the sequence $x=(x_k)$ given by

$$\Delta^{\alpha}x_k=\begin{cases}\frac{1}{\sqrt{k}}, & k\neq m^3\\ 1 & k=m^3\end{cases}. \tag{3}$$

Then, $x\in S(\Delta^{\alpha})-w_p(\Delta^{\alpha},f)$ in case $p_k=1$ and $f(x)=x$ is unbounded.

Theorem 3.3 $S\left(\Delta^{\alpha}\right)\subset w_p\left(\Delta^{\alpha}, f\right)$ where f is bounded.

Proof. Let $\varepsilon>0$, $\sum_1$ and $\sum_2$ be defined as in previous theorem. We have an integer M such that $f(x)<M$, for all $x>0$ because of f is bounded. Then

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n}\left[f\left(\left|\Delta^{\alpha}x_k-1\right|\right)\right]^{p_k} &\leq \frac{1}{n}\left(\sum_1\left[f\left(\left|\Delta^{\alpha}x_k-1\right|\right)\right]^{p_k}+\sum_2\left[f\left(\left|\Delta^{\alpha}x_k-1\right|\right)\right]^{p_k}\right)\\ &\leq \frac{1}{n}\sum_1\max\left(M^h,M^H\right)+\frac{1}{n}\sum_2\left[f(\varepsilon)\right]^{p_k}\\ &\leq \max\left(M^h,M^H\right)\frac{1}{n}\left|\left\{k\leq n : \left|\Delta^{(\alpha)}x_k-1\right|\geq\varepsilon\right\}\right|\\ &\quad+\max\left(f(\varepsilon)^h, f(\varepsilon)^H\right).\end{aligned}$$

Hence, $x\in w_p(\Delta^{\alpha}, f)$. Also from the sequence which defined in (3) $S(\Delta^{\alpha})\subset w_p(\Delta^{\alpha}, f)$ does not hold for an unbounded f.

Theorem 3.4 Let f is a bounded modulus function, we have $S(\Delta^{\alpha})=w_p(\Delta^{\alpha}, f)$.

Proof. Let f is bounded. We have the equality $S(\Delta^{\alpha})=w_p(\Delta^{\alpha}, f)$ by Theorem 3.2 and Theorem 3.3.

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