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Some Results on D-Homothetic Deformations

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Abstract

The main goal of this article is to investigate the D-homothetic deformation of an almost alpha cosymplectic manifold. Some results on almost alpha cosymplectic manifolds using D-homothetic deformation are obtained. Finally, an illustrative example is given on such manifolds.

D-Homotetik Deformasyonlar Üzerine Bazı Sonuçlar

Anahtar kelimeler
D-homotetik deformasyon; Kenmotsu manifold; Hemen hemen alfa kosimplektik manifold.

Özet

Bu makalenin asıl amacı bir hemen hemen alfa kosimplektik manifoldun D-homotetik deformasyonlarını araştırmaktır. D-homotetik deformasyon kullanılarak hemen hemen alfa kosimplektik manifoldlar üzerinde bazı sonuçlar elde edilmiştir. Son olarak, bu tür manifoldlar üzerinde bir örnek verilmiştir.

1. Introduction

An extensive researh about D-homothetic deformation on contact geometry is carried out in recent years. In 1968, a serious study in the literature was introduced by Tanno (Tanno 1968). The D-homothetic deformation is related to the following tensor structures. In other words, it means that the changing of the tensor form

$$\eta^I = b\eta, \xi^I = \left(\frac{1}{b}\right)\xi, \varphi^I = \varphi,$$

$$g^I = bg + b(b-1)\eta \otimes \eta, \quad (1)$$

where b is a positive constant (Tanno 1968). In particular, some authors studied D-homothetic deformations of such certain structures (Carriazo et al. 2011), (Montana et al. 2013), (De et al. 2013).

Using a D-homothetic deformation to an almost cosymplectic structures (φ, ξ, η, g), we have the following special condition for almost contact metric manifold

$$R(X,Y)\xi = \eta(Y)(\kappa I + \mu h + \nu\varphi h)X$$

$$-\eta(X)(\kappa I + \mu h + \nu\varphi h)Y , \quad (2)$$

for $\kappa,\mu,\nu \in R_\eta(M^{2n+1})$, where $R_\eta(M^{2n+1})$ be the subring of the ring of smooth functions f on M^{2n+1} such that $df \wedge \eta = 0$ (Olszak et al. 2005). Such manifolds are called almost cosymplectic (κ,μ,ν)-spaces. The condition (2) is invariant with respect to the D-homothetic deformations of these structures.

In this paper, we consider the D-homothetic deformations of almost alpha cosymplectic manifolds. We give some basic concepts of almost alpha cosymplectic manifolds. Next, some results are obtained related to D-homothetic deformation on almost alpha cosymplectic manifolds where alpha is a smooth function such that $d\alpha \wedge \eta = 0$. Finally, we give an example on such manifolds.

2. Preliminaries

Almost contact manifolds have odd-dimension. Let us denote the manifold that we study on by $^{2n+1}$. Then it carries two fields and a 1-form . These fields are denoted by φ and ξ. The field φ represents the endomorphisms of the tangent spaces. The field ξ is called characteristic vector field. Also, η is an 1-form given by

$$\varphi^2 = -I + \eta \otimes \xi, \eta(\xi) = 1,$$

such that $I: TM^{2n+1} \to TM^{2n+1}$ is the identity transformation. In light of the above information, it follows that

$$\varphi\xi = 0, \eta \circ \varphi = 0,$$

and the (1,1)-tensor field φ is of constant rank $2n$ (Blair 2002). Let $(M^{2n+1}, \varphi, \xi, \eta)$ be an almost contact manifold. This manifold called normal if the following tensor field N

$$N = [\varphi, \varphi] + 2d\eta \otimes \xi,$$

vanishes identically. Furthermore, $[\varphi,\varphi]$ represents the Nijenhuis tensor of the tensor field φ. It is well known that $(M^{2n+1}, \varphi, \xi, \eta)$ inducts the following Riemannian metric g

$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y), \qquad (3)$$

for arbitrary vector fields X, Y on M^{2n+1}. The above metric g is said to be a compatible metric. Thus the structure given with this quadruple called almost contact metric structure. Such manifolds are said to be the same name. According to (3), we have $\eta = g(.,\xi)$. Moreover, The Φ represents the 2-form of the manifold that is given by

$$\Phi(X,Y) = g(\varphi X, Y),$$

Then it is called the fundamental 2-form of M^{2n+1}. For an almost contact metric manifold, if both η and Φ are closed, then it is said to be an almost cosymplectic manifold. In addition, if an almost contact metric manifold holds the following equations

$$d\eta = 0, d\Phi = 2\eta \wedge \Phi,$$

Then it is called an almost Kenmotsu manifold. These manifolds are studied in (Kim et al. 2005), (Vaisman 1980), (Kenmotsu 1972) and (Olszak 1989).

where alpha is a real constant with $\alpha \neq 0$. Considering the following deformation

$$\eta^* = \left(\frac{1}{\alpha}\right)\eta, \xi^* = \alpha\xi,$$

$$\varphi^* = \varphi, g^* = \left(\tfrac{1}{\alpha^2}\right)g, \qquad (4)$$

Thus we have an almost alpha Kenmotsu structure $(\varphi^*, \xi^*, \eta^*, g^*)$. In general, this deformation is said to be a homothetic deformation (Olszak et al. 2005). The almost alpha Kenmotsu structure is connected with some local conformal deformations of almost cosymplectic structures (Vaisman 1980).

The notion defined by $d\eta = 0$ and $d\Phi = 2\alpha\eta \wedge \Phi$ called almost alpha cosymplectic manifold for arbitrary real number alpha (Kim et al. 2005).

We define $A = -\nabla\xi$ and $h = (1/2)L_\xi\varphi$ for all vector fields where alpha is a smooth function such that $d\alpha \wedge \eta = 0$ and recall that $A(\xi) = 0$ and $h(\xi) = 0$. Then we have

$$\nabla_X\xi = -\alpha\varphi^2 X - \varphi h X, \qquad (5)$$

$$(\varphi h)X + (h\varphi)X = 0, \qquad (6)$$

$$(\varphi A)X + (A\varphi)X = -2\alpha\varphi, \quad (7)$$

$$(\nabla_X \eta)Y = \alpha[g(X,Y) - \eta(X)\eta(Y)]$$
$$+g(\varphi Y, hX), \quad (8)$$

$$\delta\eta = -2\alpha n, \quad (9)$$

$$tr(h) = 0, \quad (10)$$

for arbitrary vector fields X, Y on M^{2n+1}, (Kim et al. 2005).

Let us suppose $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost alpha cosymplectic manifold. So the curvature properties are held with the aid of the Riemannian curvature tensor for arbitrary vector fields X and Y. Here alpha is a smooth function where $d\alpha \wedge \eta = 0$, and $l = R(.,\xi)\xi$ is the Jacobi operator (Öztürk et al. 2017):

$$R(X,Y)\xi = (\nabla_Y \varphi h)X - (\nabla_X \varphi h)Y - \alpha[\eta(X)\varphi hY - \eta(Y)\varphi hX] + [\alpha^2 + \xi(\alpha)][\eta(X)Y - \eta(Y)X], \quad (11)$$

$$lX = [\alpha^2 + \xi(\alpha)]\varphi^2 X + 2\alpha\varphi hX - h^2 X + \varphi(\nabla_\xi h)X, \quad (12)$$

$$lX - \varphi l \varphi X = 2[(\alpha^2 + \xi(\alpha))\varphi^2 X - h^2 X], \quad (13)$$

$$(\nabla_\xi h)X = -\varphi lX - [\alpha^2 + \xi(\alpha)]\varphi X - 2\alpha hX - \varphi h^2 X, \quad (14)$$

$$S(X,\xi) = -2n[\alpha^2 + \xi(\alpha)]\eta(X)$$
$$-\big(div(\varphi h)\big)X, \quad (15)$$

$$S(\xi,\xi) = -\big[2n\big(\alpha^2 + \xi(\alpha)\big) + tr(h^2)\big]. \quad (16)$$

3. Some Results

In this section, we consider the D-homothetic deformations of almost alpha cosymplectic manifolds with the help of (1).

Firstly, we can state the definition of D-homothetic deformation on such manifolds.

Definition 3.1. The structure of (φ, ξ, η, g) on almost cosymplectic manifold by the help of D-homothetic deformation is defined as

$$\varphi^* = \varphi, \qquad \xi^* = (1/\zeta)\xi,$$

$$\eta^* = \zeta\eta, \ g^* = \gamma g + (\zeta^2 - \gamma)\eta \otimes \eta, \quad (17)$$

where $R_\eta(M^{2n+1})$ be the subring of the smooth functions f such that $f: M \to R$ satisfying $df \wedge \eta = 0$ on M^{2n+1}. Here γ is a positive constant and $\zeta \in R_\eta((M^{2n+1}), \zeta \neq 0$ at any point of M^{2n+1} (Olszak 2005).

Now, we can give the following results:

Theorem 3.1. Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost alpha cosymplectic manifold. The manifold is transformed into a new almost ζ^*-cosymplectic manifold where alpha is parallel along ξ.

Proof. By the help of Definition 3.1, we have

$$\Phi^* = \gamma\Phi, \ d\eta^* = (d\zeta \wedge \eta) + \zeta d\eta, \quad (18)$$

where γ is positive constant and $\zeta \in R_\eta((M^{2n+1}),$ $\zeta \neq 0$ at any point of M^{2n+1}.

Follows from (18), we get

$$d\Phi^* = 2(\alpha/\zeta)(\eta^* \wedge \Phi^*), \ d\eta^* = 0, \quad (19)$$

where alpha is a smooth function such that $d\alpha \wedge \eta = 0$.

Then taking $(\alpha/\zeta) = \zeta^*$, deformed structure $(\varphi^*, \xi^*, \eta^*, g^*)$ can be written as

$$\Phi^* = \gamma\Phi, d\eta^* = 0,,$$

$$d\Phi^* = 2\zeta^*(\eta^* \wedge \Phi^*), \quad (20)$$

where $\zeta^* \in R_\eta(M^{2n+1})$. It is note that

$$X(\zeta^*) = d\zeta^*(\xi)\eta^* X$$

for any vector field X. Hence, if we choose $\xi(\alpha) = 0$, the proof is clearly completed.

Corollary 3.1. For a D-homothetic deformation, an almost α-cosymplectic structure (φ, ξ, η, g) transforms into a new almost ζ^*-cosymplectic structure $(\varphi^*, \xi^*, \eta^*, g^*)$ on the same manifold.

Theorem 3.2. Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost alpha cosymplectic manifolds. For a D-homothetic deformation of almost alpha cosymplectic structure, the Levi-Civita connections ∇^* and ∇ are can be written as follows:

$$\nabla_X Y = \nabla^*{}_X Y + \frac{(\zeta^2-\gamma)}{\zeta^2} g(AX,Y)\xi$$

$$-\frac{d\zeta(\xi)}{\zeta}\eta(X)\eta(Y)\xi. \quad (21)$$

where α is parallel along ξ.

Proof. Using Kozsul's formula we have

$$2g^*(\nabla^*{}_X Y, Z) = Xg^*(Y,Z) + Yg^*(X,Z) - Zg^*(X,Y) + g^*([X,Y],Z) + g^*([Z,X],Y) + g^*([Z,Y],X) \quad (22)$$

for any vector fields X, Y, Z.

Considering the following equation

$$g^* = \gamma g + (\zeta^2 - \gamma)\eta \otimes \eta$$

with all components of Kozsul's formula, then we get

$$2g^*(\nabla^*{}_X Y, Z) = 2\gamma g(\nabla_X Y, Z) + 2\zeta d\zeta(\xi)\eta(X)\eta(Y)\eta(Z) + (\zeta^2 - \gamma)[2\eta(\nabla_X Y)\eta(Z) + 2g(Y, \nabla_X \xi)\eta(Z)], \quad (23)$$

for $\xi(\alpha) = 0$. Moreover, we have

$$2g^*(\nabla^*{}_X Y, Z) = 2\gamma g(\nabla^*{}_X Y, Z))$$

$$+2(\zeta^2 - \gamma)\eta(\nabla^*{}_X Y)\eta(Z). \quad (24)$$

From (23) and (24), we obtain

$$\gamma g(\nabla^*{}_X Y, Z) + (\zeta^2 - \gamma)\eta(\nabla^*{}_X Y)\eta(Z) = \gamma g(\nabla_X Y, Z) + \zeta d\zeta(\xi)\eta(X)\eta(Y)\eta(Z) + (\zeta^2 - \gamma)\eta(\nabla_X Y)\eta(Z) + g(Y, \nabla_X \xi)\eta(Z), \quad (25)$$

such that

$$\eta(\nabla^*{}_X Y) = (1/\zeta)d\zeta(\xi)\eta(X)\eta(Y) + \eta(\nabla_X Y)$$

$$+(\zeta^2 - \gamma)/\zeta))g(Y, \nabla_X \xi). \quad (26)$$

Next, taking into account of (25) and (26), we have

$$g(\nabla^*{}_X Y, Z) = g(\nabla_X Y, Z) + (1/\zeta)d\zeta(\xi)\eta(X)\eta(Y)\eta(Z) \quad (27)$$

$$+(\zeta^2 - \gamma)/\zeta)(1 - (\zeta^2 - \gamma)/\zeta)g(Y, \nabla_X \xi)\eta(Z).$$

Therefore, arranging the above equation, it holds (21). Thus the proof is completed.

Theorem 3.3. For a D-homothetic deformation of almost alpha cosymplectic structure, the following equations are held

$$A^*X = (1/\zeta)AX, h^*X = (1/\zeta)hX, \quad (28)$$

$$R^*(X,Y)\xi^* = (1/\zeta)R(X,Y)\xi$$

$$+(1/\zeta^2)d\zeta(\xi)[\eta(X)AY - \eta(Y)AX], \quad (29)$$

for any vector fields X, Y, Z and $\xi(\alpha) = 0$.

Proof. First, from (5), (6), (18) and (20), we have

$$A^*X = \left(\frac{X(\zeta)}{\zeta^2}\right)\xi - \left(\frac{1}{\zeta}\right)\nabla_X \xi$$

$$-\left(\frac{1}{\zeta^2}\right)d\zeta(\xi)\eta(X)\xi, \quad (30)$$

where α is parallel along ξ.

Making use of (18) and (30) it follows that (28) is obvious.

Next, in order to prove (29), let us consider the Riemannian curvature tensor and (18). In other words, we have

$$R^*(X,Y)\xi^* = \nabla^*{}_X \nabla^*{}_Y \xi^* - \nabla^*{}_Y \nabla^*{}_X \xi^* - \nabla^*{}_{[X,Y]}\xi^*$$

$$= -\left(\frac{X(\zeta)}{\zeta^2}\right)\nabla_Y \xi - \left(\frac{Y(\zeta)}{\zeta^2}\right)\nabla_X \xi + \left(\frac{1}{\zeta}\right)\nabla^*{}_X \nabla_Y \xi - \left(\frac{1}{\zeta}\right)\nabla^*{}_Y \nabla_X \xi - \left(\frac{1}{\zeta}\right)\nabla_{[X,Y]}\xi. \quad (31)$$

Then simplifying the above equation, we get

$$R^*(X,Y)\xi^* = \left(\frac{X(\zeta)}{\zeta^2}\right)AY - \left(\frac{Y(\zeta)}{\zeta^2}\right)AX$$

$$+\left(\frac{1}{\zeta}\right)R(X,Y)\xi, \quad (32)$$

which is a consequence of (21) and (28). Here it is noticed that we use the following formula for the desired result

$$\nabla^*{}_X\nabla_Y\xi = \nabla_X\nabla_Y\xi + \left(\frac{\zeta^2-\gamma}{\zeta^2}\right) g(Y, A^2X)\xi. \qquad (33)$$

Example 3.1. We suppose the standart coordinates of $R^{2n+1}(x_1, \dots, x_n, y_1, \dots, y_n, z)$ and consider (2n+1)-dimensional manifold defined by

$$M^* = \{(x_1, \dots, x_n, y_1, \dots, y_n, z) \in R^{2n+1}\colon\ z \neq 0\}.$$

Also, we choose the global basis of M^*:

$$X_i = 2z\left(\frac{\partial}{\partial x_i}\right), \qquad Y_i = -\left(\frac{2}{z^3}\right)\left(\frac{\partial}{\partial y_i}\right),$$

$$\xi = \left(\frac{\partial}{\partial z}\right), \qquad i = 1,2, \dots, n$$

Also, we put

$$g = \sum_{i=1}^{n}\left(\frac{1}{4}\right)\left(\left(\frac{1}{z^2}\right)dx_i^2 + z^6 dy_i^2\right) + dz^2,$$

$$\eta = dz,$$

and

$$\varphi(\partial/(\partial x_i) = -(1/(z^4))(\partial/(\partial y_i),$$

$$\varphi((\partial/(\partial y_i) = z^4(\partial/(\partial x_i),\ \ \varphi(\xi) = 0,$$

Here it is clear that (φ, ξ, η, g) is an almost contact structure on M^*.

So we verify the condition $d\Phi = 2\alpha(\eta \wedge \Phi)$, where all Φ_{ij}'s vanish except for

$$\Phi_{ii} = g(\varphi((\partial/(\partial y_i))),\ (\partial/(\partial x_i))) = (z^2/4).$$

Moreover, we have

$$d\Phi = \left(\frac{1}{2z}\right)(dx \wedge dy \wedge dz)$$

$$= \left(\frac{1}{8z}\right)(\eta \wedge \Phi),$$

such that

$$\Phi = (z^2/4)\sum_{i=1}^{n}(dx_i \wedge dy_i),$$

for the exterior derivative.

Thus we get the smooth function defined by

$$\alpha(z) = \left(\frac{1}{16z}\right).$$

Finally, the structure (φ, ξ, η, g) is an almost alpha cosymplectic one. It is clear that N_φ does not vanish.

4. Discussion and Conclusion

In this paper, we are especially interested in almost alpha cosymplectic manifolds in the light of (1). Some certain results are obtained related to D-homothetic deformation on almost alpha cosymplectic manifolds where alpha is a smooth function such that $d\alpha \wedge \eta = 0$.

Our forthcoming paper is devoted to investigate almost alpha cosymplectic (κ, μ, ν)-spaces in terms of a certain D-homothetic deformation given in this paper. Open problems are so interesting for these spaces where the smooth functions κ, μ, ν are not constants.

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