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Building A Different Family of Nullnorms on Lattices

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Anahtar kelimeler
Bounded lattice;
Nullnorm;
Triangular norm;
Triangular conorm.

Öz

In this paper, we introduce a new method for obtaining nullnorms on M having an annihilator based on the existence of triangular norms and nullnorms acting on a subinterval of a bounded lattice M. Some basic properties concerning this construction method are investigated. Furthermore, it is exemplified the fact that the proposed method differs from the existing methods for constructing nullnorms on bounded lattices.

Kafesler Üzerinde Nullnormların Bir Farklı Ailesinin İnşaası

Keywords
Sınırlı kafes;
Nullnorm;
Üçgensel norm;
Üçgensel Konorm.

Abstract

Bu çalışmada sınırlı bir M kafesinin bir alt aralığı üzerinde hareket eden nullnormlar ve üçgensel normlar temel alınarak M üzerinde bir yutan elemanlı nullnormları elde etmek için yeni bir inşaa yöntemi önerilmektedir. Bu inşaa yöntemi ile ilgili bazı temel özellikler araştırılmaktadır. Ayrıca, önerilen yöntemin sınırlı kafesler üzerinde nullnormları inşa eden mevcut yöntemlerden farklı olduğu örnekle gösterilmektedir.

1. Introduction

The definitions of nullnorms and t-operators first appeared in (Calvo et al. 2001, Mas et al. 1999), respectively. Both of these operators are special aggregation operators, that are demonstrated beneficial in numerous areas like aggregation, expert systems, neural networks, multicriteria decision support, fuzzy logic and fuzzy system modeling (Calvo et al. 2001, Dubois and Prade 2000, Klement et al. 2000, Takács 2008). In particular, these operators are used in fuzzy logic as aggregation operators *"or"* and maintain several logical characteristics. Nullnorms as generalizations of the notions of triangular norms and triangular conorms, admit an annihilator to be any element from unit interval. Furthermore, the fact that t-operators and nullnorms are equivalent in (Mas et al. 2002) were proved. That is, whenever a mapping is a t-operator, it is a nullnorm and on the other hand, once a mapping is nullnorm, it is a t-operator. There are several approaches dealing with nullnorms on the real unit interval in the papers (Drygaś, 2004a, Drygaś et al. 2017, Grabisch 2009, Sun et al. 2017).

Nullnorms on general bounded lattices were studied in (Karaçal et al. 2015). It was showed the fact that a nullnorm defined on a general bounded lattice having an annihilator with underlying triangular norms and triangular conorms always exists. Moreover, some construction methods for such nullnorms were introduced. By means of these methods, the greatest and the smallest nullnorms on general bounded lattices were obtained. In (Ertuğrul 2018), elaborating the mappings introduced previously by (Karaçal et al. 2015) and enhancing them, further methods for generating nullnorms defined on general bounded lattices were described.

In this study, we present a new method to generate nullnorms defined on a bounded lattice M having the fixed annihilator via the existence of a triangular norm and a nullnorm on a subinterval of M. In addition, some main characteristics of the proposed methods are researched. The paper includes three parts. In Section 2, some main results about nullnorms defined on bounded lattices are given. In Section 3, it is described a method for generating nullnorms defined on a bounded lattice different from previously proposed methods. It is also demonstrated the fact that our method differs from the existing methods. Some illustrative examples are provided in order to show the feasibility of our construction method.

2. Preliminaries

In this part, some main results dealing with bounded lattices and nullnorms (triangular norms, triangular conorms) defining on them are given.

Definition 1 (Birkhoff 1967). A partially ordered set $(P, \leq)$ is called lattice if any two elements a, b in P have the greatest lower bound denoted by $\inf\{a, b\}$ or $a \wedge b$ and the least upper bound denoted by $\sup\{a, b\}$ or $a \vee b$.

A lattice $(M, \leq)$ is bounded whenever M has the bottom element represented as 0_M and top element represented as 1_M (i.e. there are two elements $1_M, 0_M \in M$ such that for all $a \in M$, $0_M \leq a \leq 1_M$).

Throughout this paper, M always represents any given general bounded lattice with the top element 1_M and bottom element 0_M unless otherwise stated.

Definition 2 (Birkhoff 1967). Let $p, q \in M$. In that case p and q are incomparable, it is used the notation $f \parallel f$.

The set of elements the fact that are incomparable with $s \in M$ is denoted by I_s. So, $I_s = \{p \in M \mid p \parallel s\}$.

Definition 3 (Birkhoff 1967). Let $p, q \in M$ and $f \leq f$. Then it is defined a subinterval $[p, q]$ in M as below:

$$[p, q] = \{x \in M \mid p \leq x \leq q\}.$$

Similarly, it can be defined the other subintervals in M as follow

$]p, q] = \{x \in M \mid p < x \leq q\}$,

$[p, q[= \{x \in M \mid p \leq x < q\}$

and

$]p, q[= \{x \in M \mid p < x < q\}$.

Definition 4 (Çaylı and Karaçal 2018a and Karaçal et al. 2015). If a mappping $R: M^2 \to M$ is associative, commutative monotone and there is an element $s \in M$ such that $R(a, 0_M) = a$ for all $a \leq s$ and $R(a, 1_M) = a$ for all $a \geq s$ then it is called a nullnorm on M.

It is obvious the fact that for all $a \in M$, $R(a, s) = s$. So, $s \in M$ is an annhilator, i.e., absorbing element or zero element of R.

Definition 5 (Çaylı et al. 2016, Çaylı 2018). If a mapping $T: M^2 \to M$ having an annihilator 0 is associative, commutative, monotone, then it is called a triangular norm on M.

Definition 6 (Çaylı et al. 2016, Çaylı 2018). If a mapping $S: M^2 \to M$ having an annihilator 1_M is associative, commutative, monotone, then it is called a triangular norm on M.

Example 1. The least triangular norm $T_\wedge: M^2 \to M$ and the greatest triangular norm $T_W: M^2 \to M$, respectively, are defined as:

$$T_W(a, b) = \begin{cases} b & \text{if } a = 1_M, \\ a & \text{if } b = 1_M, \\ 0_M & \text{otherwise.} \end{cases}$$

$$T_\wedge(a, b) = \inf\{a, b\}$$

The least triangular conorm $S_\vee: M^2 \to M$ and the greatest triangular conorm $S_W: M^2 \to M$, respectively, are defined as:

$$S_\vee(a,b) = \sup\{a,b\}$$

$$S_W(a,b) = \begin{cases} b & \text{if } a = 0_M, \\ a & \text{if } b = 0_M, \\ 1_M & \text{otherwise.} \end{cases}$$

The following set is denoted by D_s:
$D_s = [0_M, s] \times [s, 1_M] \cup [s, 1_M] \times [0_M, s]$
for $s \in M \backslash \{0_M, 1_M\}$.

Proposition 1 (Drygaś 2004b, Çaylı and Karaçal 2018b, Karaçal et al. 2016). Consider the fact that $s \in M$ and $s \neq 0_M, 1_M$, and a nullnorm R having the annihilator s. The following statements hold:

(i) $R \upharpoonright [0_M, s]^2 : [0_M, s]^2 \to [0_M, s]$ is a triangular conorm on the subinterval $[0_M, s]$.

(ii) $R \upharpoonright [s, 1_M]^2 : [s, 1_M]^2 \to [s, 1_M]$ is a triangular norm on the subinterval $[s, 1_M]$.

Proposition 2 (Drygaś 2004b, Karaçal et al. 2015). Consider the fact that $s \in M$ and $s \neq 0_M, 1_M$, and a nullnorm R having the annihilator s. The following statements hold:

(i) $R(a,b) = s$ for $(a,b) \in D_s$.

(ii) $s \leq R(a,b)$ for $(a,b) \in [s, 1_M]^2 \cup [s, 1_M] \times I_s \cup I_s \times [s, 1_M]$.

(iii) $R(a,b) \leq s$ for $(a,b) \in [0_M, s]^2 \cup [0_M, s] \times I_s \cup I_s \times [0_M, s]$.

(iv) $R(a,b) \leq b$ for $(a,b) \in M \times [s, 1_M]$.

(v) $R(a,b) \leq a$ for $(a,b) \in [s, 1_M] \times M$.

(vi) $a \leq R(a,b)$ for $(a,b) \in [0_M, s] \times M$.

(vii) $b \leq R(a,b)$ for $(a,b) \in M \times [0_M, s]$.

(viii) $\sup\{a,b\} \leq R(a,b)$ for $(a,b) \in [0_M, s]^2$.

(ix) $R(a,b) \leq \inf\{a,b\}$ for $(a,b) \in [s, 1_M]^2$.

(x) $\sup\{\inf\{a,s\}, \inf\{b,s\}\} \leq R(a,b)$
for $(a,b) \in [0_M, s] \times I_s \cup I_s \times [0_M, s] \cup I_s \times I_s$.

(xi) $R(a,b) \leq \inf\{\sup\{a,s\}, \sup\{b,s\}\}$
for $(a,b) \in [s, 1_M] \times I_s \cup I_s \times [s, 1_M] \cup I_s \times I_s$.

2. Construction methods for nullnorms

After the demonstration of the presence of nullnorms defined on M, the constructions for nullnorms on M have recently attracted much attention. In the literature, there are some construction methods for generating nullnorms defined on M. In (Karaçal et al. 2015, Ertuğrul 2018), it was presented three methods for obtaining nullnorms on M having the fixed annihilator $s \in M$ the fact that $s \neq 0_M, 1_M$ based on the presence of triangular norms on $[s, 1_M]$ and triangular conorms on $[0_M, s]$.

These construction methods in Theorem 1 and Theorem 2 are recalled.

Theorem 1 (Karaçal et al. 2015). Consider the fact that $s \in M$ and $s \neq 0_M, 1_M$, a triangular conorm S acting on $[0_M, s]$ and a triangular norm T acting on $[s, 1_M]$. in the present case, the mapping $R^{(T,S)}: M^2 \to M$ is a nullnorm on M having the annihilator s, where

$$R^{(T,S)}(a,b) = \begin{cases} S(a,b) & \text{if } (a,b) \in [0_M, s]^2, \\ T(a,b) & \text{if } (a,b) \in [s, 1_M]^2, \\ s & \text{otherwise.} \end{cases}$$

Theorem 2 (Ertuğrul 2018). Consider the fact that $s \in M$ and $s \neq 0_M, 1_M$, a triangular conorm S acting on $[0_M, s]$ and a triangular norm T acting on $[s, 1_M]$. In this case the mappings $R_T^S, R_S^T: M^2 \to M$ are nullnorms on M having the annihilator s, where

$$R_T^S(a,b) = \begin{cases} S(a,b) & \text{if } (a,b) \in [0_M, s]^2, \\ T(a,b) & \text{if } (a,b) \in [s, 1_M]^2, \\ S(\inf\{a,s\}, \inf\{b,s\}) & \begin{array}{l}\text{if } (a,b) \in [0_M, s] \times I_s \\ \cup I_s \times [0_M, s] \cup I_s \times I_s\end{array} \\ s & \text{otherwise.} \end{cases}$$

and

$$R_S^T(a,b) = \begin{cases} S(a,b) & \text{if } (a,b) \in [0_M,s]^2, \\ T(a,b) & \text{if } (a,b) \in [s,1_M]^2, \\ T(\sup\{a,s\},\sup\{b,s\}) & \text{if } (a,b) \in [s,1_M] \times \\ & \cup I_s \times [s,1_M] \cup I_s \times \\ s & \text{otherwise.} \end{cases}$$

Corollary 1 (Karaçal et al. 2015). Consider the fact that $s \in M$ and $s \neq 0_M, 1_M$, a triangular conorm S acting on $[0_M, s]$ and a triangular norm T acting on $[s, 1_M]$. In the present case, the mappings $R^{(\vee)}, R^{(\wedge)}: M^2 \to M$ are the greatest nullnorm and the least nullnorm on M having the annihilator s, respectively, where

$$R^{(\vee)}(a,b) = \begin{cases} \sup\{a,b\} & \text{if } (a,b) \in [0_M,s]^2, \\ s & \text{if } (a,b) \in [s,1_M[^2 \cup [s,1_M] \times I_s \\ & \cup I_s \times [s,1_M] \cup D_s, \\ \sup\{\inf\{a,s\},\inf\{b,s\}\} & \text{if } (a,b) \in [0_M,s] \times I_s \\ & \cup I_s \times [0_M,s] \cup I_s \times I_s, \\ \inf\{a,b\} & \text{otherwise.} \end{cases}$$

and

$$R^{(\wedge)}(a,b) = \begin{cases} \inf\{a,b\} & \text{if } (a,b) \in [s,1_M]^2, \\ s & \text{if } (a,b) \in [0_M,s[\times I_s \\ & \cup I_s \times [0_M,s] \cup D_s, \\ \inf\{\sup\{a,s\},\sup\{b,s\}\} & \text{if } (a,b) \in [s,1_M] \times I_s \\ & \cup I_s \times [s,1_M] \cup I_s \times I_s, \\ \sup\{a,b\} & \text{otherwise.} \end{cases}$$

Corollary 2. Consider the fact that $s \in M$ and $s \neq 0_M, 1_M$, a triangular conorm S on acting $[0_M, s]$ and a triangular norm T acting on $[s, 1_M]$. In Theorem 2, the nullnorms $R_T^S, R_S^T: M^2 \to M$ can also be given by

$$R_T^S(a,b) = \begin{cases} S(a,b) & \text{if } (a,b) \in [0_M,s]^2, \\ T(a,b) & \text{if } (a,b) \in [s,1_M]^2, \\ S(\inf\{a,s\},\inf\{b,s\}) & \text{otherwise.} \end{cases}$$

and

$$R_S^T(a,b) = \begin{cases} T(a,b) & \text{if } (a,b) \in [s,1_M]^2, \\ S(a,b) & \text{if } (a,b) \in [0_M,s]^2, \\ T(\sup\{a,s\},\sup\{b,s\}) & \text{otherwise.} \end{cases}$$

In order to enhance the methods proposed in (Karaçal et al. 2015, Ertuğrul 2018), it is introduced a new construction method for nullnorms on M having the annihilator $s \in M$ such that $s \neq 0_M, 1_M$ different from the proposal in (Karaçal et al. 2015, Ertuğrul 2018). Our methods base on the presence of a nullnorm acting on a subinterval $[0_M, f]$ of M and a triangular norm on $[f, 1_M]$ for any element $f \in M \backslash \{0_M, 1_M\}$. In addition, our construction method differs from the methods presented by (Karaçal et al. 2015, Ertuğrul 2018) and they are described in Theorem 1 and Theorem 2.

Theorem 3. Consider the fact that $s, f \in M$ and $s, f \neq 0_M, 1_M$ and $s \in]0_M, f]$. If R_f is a nullnorm on $[0_M, f]$ having the annihilator s and T_f is a triangular norm acting on $[f, 1_M]$, in this case the following mapping $R_c: M^2 \to M$ is a nullnorm on M having the annihilator s.

$$R_c(p,q) = \begin{cases} R_f(p,q) & \text{if } (p,q) \in [0_M,f]^2, \\ T_f(p,q) & \text{if } (p,q) \in [f,1_M]^2, \\ R_f(\inf\{p,f\},\inf\{q,f\}) & \text{otherwise.} \end{cases}$$

Proof. i) Monotonicity: It is proved the fact that if $p \leq q$ in this case, $R_c(p,z) \leq R_c(q,z)$ for all $z \in M$. It is proved considering all possible cases.

1. Let $p \in [0_M, f]$. In the present case,
   - If $q \in [0_M, f]$, then
   - If $z \in [0_M, f]$, then

$$R_c(p,z) = R_f(p,z) \leq R_f(q,z) = R_c(q,z)$$

   - If $z \in]f, 1_M]$, then

$$R_c(p,z) = R_f(p,f) \leq R_f(q,f) = R_c(q,z)$$

   - If $z \in I_f$, then

$$\begin{aligned} R_c(p,z) &= R_f(p,\inf\{z,f\}) \\ &\leq R_f(q,\inf\{z,f\}) \\ &= R_c(q,z) \end{aligned}$$

   - If $q \in]f, 1_M]$, then
   - If $z \in [0_M, f]$, then

$$R_c(p,z) = R_f(p,z) \leq R_f(f,z) = R_c(q,z)$$

   - If $z \in]f, 1_M]$, then

$$R_c(p,z) = R_f(p,f) \leq f \leq T_f(q,z) = R_c(q,z)$$

   - If $z \in I_f$, then

$$\begin{aligned} R_c(p,z) &= R_f(p,\inf\{z,f\}) \\ &\leq R_f(f,\inf\{z,f\}) \\ &= R_c(q,z) \end{aligned}$$

   - If $q \in I_f$,
   - If $z \in [0_M, f]$, then

$R_c(p,z) = R_f(p,z) \leq R_f(\inf\{q,f\},z) = R_c(q,z)$

- If $z \in]f, 1_M]$, then

$R_c(p,z) = R_f(p,f)$
$\leq R_f(\inf\{q,f\},f)$
$= R_c(q,z)$

- If $z \in I_f$, then

$R_c(p,z) = R_f(p,\inf\{z,f\})$
$\leq R_f(\inf\{q,f\},\inf\{z,f\})$
$= R_c(q,z)$

2. Let $p \in]f, 1_M]$. In the present case, $q \in]f, 1_M]$.

- If $z \in [0_M, f]$, then

$R_c(p,z) = R_f(f,z) = R_c(q,z)$

- If $z \in]f, 1_M]$, then

$R_c(p,z) = T_f(p,z) \leq T_f(q,z) = R_c(q,z)$

- If $z \in I_f$, then

$R_c(p,z) = R_f(f,\inf\{z,f\}) = R_c(q,z)$

3. Let $p \in I_f$. In the present case,

- If $q \in]f, 1_M]$, then
- If $z \in [0_M, f]$, then

$R_c(p,z) = R_f(\inf\{p,f\},z) \leq R_f(f,z) = R_c(q,z)$

- If $z \in]f, 1_M]$,

$R_c(p,z) = R_f(\inf\{p,f\},f) \leq T_f(q,z) = R_c(q,z)$

- If $z \in I_f$, then

$R_c(p,z) = R_f(\inf\{p,f\},\inf\{z,f\})$
$\leq R_f(f,\inf\{z,f\})$
$= R_c(q,z)$

- If $q \in I_f$, then
- If $z \in [0_M, f]$, then

$R_c(p,z) = R_f(\inf\{p,f\},z)$
$\leq R_f(\inf\{q,f\},z)$
$= R_c(q,z)$

- If $z \in]f, 1_M]$, then

$R_c(p,z) = R_f(\inf\{p,f\},f)$
$\leq R_f(\inf\{q,f\},f)$
$= R_c(q,z)$

- If $z \in I_f$, then

$R_c(p,z) = R_f(\inf\{p,f\},\inf\{z,f\})$
$\leq R_f(\inf\{q,f\},\inf\{z,f\})$
$= R_c(q,z)$

ii) Associativity: It is proved the fact that for all $p,q,z \in M$, $R_c(p,R_c(q,z)) = R_c(R_c(p,q),z)$.
Again it is proved considering all possible cases.

1. Let $p \in [0_M, f]$. In the present case,

- If $q \in [0_M, f]$, then
- If $z \in [0_M, f]$, then

$R_c(p,R_c(q,z)) = R_c(p,R_f(q,z))$
$= R_f\left(p,R_f(q,z)\right)$
$= R_f(R_f(p,q),z)$
$= R_c(R_f(p,q),z)$
$= R_c(R_c(p,q),z)$

- If $z \in]f, 1_M]$, then

$R_c(p,R_c(q,z)) = R_c(p,R_f(q,f))$
$= R_f(p,R_f(q,f))$
$= R_f(R_f(p,q),f)$
$= R_c(R_f(p,q),z)$
$= R_c(R_c(p,q),z)$

- If $z \in I_f$, then

$R_c(p,R_c(q,z)) = R_c(p,R_f(q,\inf\{z,f\}))$
$= R_f(p,R_f(q,\inf\{z,f\}))$
$= R_f(R_f(p,q),\inf\{z,f\})$
$= R_c(R_f(p,q),z)$
$= R_c(R_c(p,q),z)$

- If $q \in]f, 1_M]$, then
- If $z \in [0_M, f]$, then

$R_c(p,R_c(q,z)) = R_c(p,R_f(f,z))$
$= R_f(p,R_f(f,z))$
$= R_f(R_f(p,f),z)$
$= R_c(R_f(p,f),z)$
$= R_c(R_c(p,q),z)$

- If $z \in]f, 1]$, then

$R_c(p,R_c(q,z)) = R_c(p,T_f(q,z))$
$= R_f(p,f)$
$= R_f(R_f(p,f),f)$
$= R_c(R_f(p,f),z)$
$= R_c(R_c(p,q),z)$

- If $z \in I_f$, then

$R_c(p,R_c(q,z)) = R_c(p,R_f(f,\inf\{z,f\}))$
$= R_f(p,R_f(f,\inf\{z,f\}))$
$= R_f(R_f(p,f),\inf\{z,f\})$
$= R_f(R_f(p,f),f)$
$= R_c(R_c(p,q),z)$

- If $q \in I_f$, then
- If $z \in [0_M, f]$,

$R_c(p,R_c(q,z)) = R_c(p,R_f(\inf\{q,f\},z))$
$= R_f(p,R_f(\inf\{q,f\},z))$
$= R_f(R_f(p,\inf\{q,f\}),z)$
$= R_c(R_f(p,\inf\{q,f\}),z)$
$= R_c(R_c(p,q),z)$

- If $z \in]f, 1_M]$, then

$R_c(p,R_c(q,z)) = R_c(p,R_f(\inf\{q,f\},f))$

$= R_f(p, R_f(\inf\{q,f\}, f))$
$= R_f(R_f(p, \inf\{q,f\}), f)$
$= R_c(R_f(p, \inf\{q,f\}), z)$
$= R_c(R_c(p,q), z)$

- If $z \in I_f$, then

$R_c(p, R_c(q,z)) = R_c(p, R_f(\inf\{q,f\}, \inf\{z,f\}))$
$= R_f(p, R_f(\inf\{q,f\}, \inf\{z,f\}))$
$= R_f(R_f(p, \inf\{q,f\}), \inf\{z,f\})$
$= R_c(R_f(p, \inf\{q,f\}), z)$
$= R_c(R_c(p,q), z)$

2. Let $p \in]f, 1_M]$. In the present case,

- If $q \in [0_M, f]$, then
- If $z \in [0_M, f]$, then

$R_c(p, R_c(q,z)) = R_c(p, R_f(q,z))$
$= R_f(f, R_f(q,z))$
$= R_f(R_f(f,q), z)$
$= R_c(R_f(f,q), z)$
$= R_c(R_c(p,q), z)$

- If $z \in]f, 1_M]$, then

$R_c(p, R_c(q,z)) = R_c(p, R_f(q,f))$
$= R_f(f, R_f(q,f))$
$= R_f(R_f(f,q), f)$
$= R_c(R_f(f,q), z)$
$= R_c(R_c(p,q), z)$

- If $z \in I_f$, then

$R_c(p, R_c(q,z)) = R_c(p, R_f(q, \inf\{z,f\}))$
$= R_f\left(f, R_f(q, \inf\{z,f\})\right)$
$= R_f(R_f(f,q), \inf\{z,f\})$
$= R_c(R_f(f,q), z)$
$= R_c(R_c(p,q), z)$

- If $q \in]f, 1_M]$, then
- If $z \in [0_M, f]$, then

$R_c(p, R_c(q,z)) = R_c\left(p, R_f(f,z)\right)$
$= R_f(f, R_f(f,z))$
$= R_c(R_f(f,f), z)$
$= R_c(R_c(p,q), z)$

- If $z \in]f, 1_M]$, then

$R_c(p, R_c(q,z)) = R_c(p, T_f(q,z))$
$= R_f(f,f)$
$= R_f(R_f(f,f), z)$
$= R_c(R_f(p,q), z)$
$= R_c(R_c(p,q), z)$

- If $z \in I_f$, then

$R_c(p, R_c(q,z)) = R_c(p, R_f(f, \inf\{z,f\}))$
$= R_f(f, R_f(f, \inf\{z,f\}))$
$= R_f\left(R_f(f,f), \inf\{z,f\}\right)$
$= R_c(R_f(f,f), z)$
$= R_c(R_c(p,q), z)$

- If $q \in I_f$, then
- If $z \in [0_M, f]$, then

$R_c(p, R_c(q,z)) = R_c(p, R_f(\inf\{q,f\}, z))$
$= R_f(f, R_f(\inf\{q,f\}, z))$
$= R_f(R_f(f, \inf\{q,f\}), z)$
$= R_c(R_f(f, \inf\{q,f\}), z)$
$= R_c(R_c(p,q), z)$

- If $z \in]f, 1_M]$, then

$R_c(p, R_c(q,z)) = R_c(p, R_f(\inf\{q,f\}, f))$
$= R_f\left(f, R_f(\inf\{q,f\}, f)\right)$
$= R_f(R_f(f, \inf\{q,f\}), f)$
$= R(R_f(f, \inf\{q,f\}), z)$
$= R_c(R_c(p,q), z)$

- If $z \in I_f$, then

$R_c(p, R_c(q,z)) = R_c(p, R_f(\inf\{q,f\}, \inf\{z,f\}))$
$= R_f\left(f, R_f(\inf\{q,f\}, \inf\{z,f\})\right)$
$= R_f\left(R_f(f, \inf\{q,f\}), \inf\{z,f\}\right)$
$= R_c(R_c(p,q), z)$

3. Let $p \in I_f$. In the present case,

- If $q \in [0_M, f]$, then
- If $z \in [0_M, f]$, then

$R_c(p, R_c(q,z)) = R_c(p, R_f(q,z))$
$= R_f(\inf\{p,f\}, R_f(q,z))$
$= R_f(R_f(\inf\{p,f\}, q), z)$
$= R_c(R_f(\inf\{p,f\}, q), z)$
$= R_c(R_c(p,q), z)$

- If $z \in]f, 1_M]$, then

$R_c(p, R_c(q,z)) = R_c(p, R_f(q,f))$
$= R_f(\inf\{p,f\}, R_f(q,f))$
$= R_f(R_f(\inf\{p,f\}, q), f)$
$= R_c(R_f(\inf\{p,f\}, q), z)$
$= R_c(R_c(p,q), z)$

- If $z \in I_f$, then

$R_c(p, R_c(q,z)) = R_c(p, R_f(q, \inf\{z,f\}))$
$= R_f(\inf\{p,f\}, R_f(q, \inf\{z,f\}))$
$= R_f\left(R_f(\inf\{p,f\}, q), \inf\{z,f\}\right)$
$= R_c(R_f(\inf\{p,f\}, q), z)$
$= R_c(R_c(p,q), z)$

- If $q \in]f, 1_M]$, then
- If $z \in [0_M, f]$, then

$R_c(p, R_c(q,z)) = R_c(p, R_f(f,z))$

$= R_f\left(\inf\{p,f\}, R_f(f,z)\right)$
$= R_f(R_f(\inf\{p,f\},f),z)$
$= R_c(R_f(\inf\{p,f\},f),z)$
$= R_c(R_c(p,q),z)$

- If $z \in]f,1_M]$, then

$R_c(p,R_c(q,z)) = R_c(p,T_f(q,z))$
$= R_f(\inf\{p,f\},f)$
$= R_f(\inf\{p,f\},R_f(f,f))$
$= R_f(R_f(\inf\{p,f\},f),f)$
$= R_c(R_c(p,q),z)$

- If $z \in I_f$, then

$R_c(p,R_c(q,z)) = R_c(p,R_f(f,\inf\{z,f\}))$
$= R_f\left(\inf\{p,f\}, R_f(f,\inf\{z,f\})\right)$
$= R_f(R_f(\inf\{p,f\},f))$
$= R_c(R_c(p,q),z)$

- If $q \in I_f$, then
- If $z \in [0_M,f]$, then

$R_c(p,R_c(q,z)) = R_c(p,R_f(\inf\{q,f\},z))$
$= R_f(\inf\{p,f\},R_f(\inf\{q,f\},z))$
$= R_f(R_f(\inf\{p,f\},\inf\{q,f\}),z)$
$= R_c(R_f(\inf\{p,f\},\inf\{q,f\}),z)$
$= R_c(R_c(p,q),z)$

- If $z \in]f,1_M]$, then

$R_c(p,R_c(q,z)) = R_c(p,R_f(\inf\{q,f\},f))$
$= R_f(\inf\{p,f\},R_f(\inf\{q,f\},f))$
$= R_f(R_f(\inf\{p,f\},\inf\{q,f\}),f)$
$= R_c(R_f(\inf\{p,f\},\inf\{q,f\}),z)$
$= R_c(R_c(p,q),z)$

- If $z \in I_f$, then

$R_c(p,R_c(q,z)) = R_c(p,R_f(\inf\{q,f\},\inf\{z,f\}))$
$= R_f\left(\inf\{p,f\}, R_f(\inf\{q,f\},\inf\{z,f\})\right)$
$= R_f(R_f(\inf\{p,f\},\inf\{q,f\}),\inf\{z,f\})$
$= R_c(R_f(\inf\{p,f\},\inf\{q,f\}),z)$
$= R_c(R_c(p,q),z)$

It is obvious the fact that R_c is commutative and s is an annihilator of R_c.

Remark 1. Consider the fact that $s,f \in M$ and $s,f \neq 0_M,1_M$ and $s \in]0_M,f]$, a nullnorm R_f on $[0_M,f]$ having the annihilator s and a triangular norm T_f acting on $[f,1_M]$.

i) If $I_s = \emptyset$, in this case the nullnorm R_c given in Theorem 3 coincides with $R^{(T,S)}$, R_T^S, R_S^T given in Theorem 1 and Theorem 2.

ii) If $f = s$, in this case the nullnorm R_c given in Theorem 3 coincides with R_T^S given in Theorem 2.

iii) If f is an atom, in this case $f = s$. From (ii) the nullnorm R_c given in Theorem 3 coincides with R_T^S given in Theorem 2.

iv) The nullnorm R_c on any bounded lattice having the annihilator s does not have to coincide with the nullnorms $R^{(T,S)}$, R_S^T, R_T^S unless some special conditions are specified. Let demonstrate in the following example, this argument:

Example 2. Take the lattice $M = \{0,s,b,c,f,1\}$ with Hasse diagram shown in Figure 1.

Given the mapping R_f on $[0_M,f]$ as given in Table 1. It is possible to check the fact that R_f is a nullnorm on $[0_M,f]$ having the annihilator s.

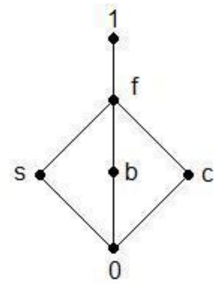

Figure 1. The lattice M

Table 1: The nullnorm R_f on $[0_M,f]$

R_m	0_M	s	b	c	f
0_M	0_M	s	0_M	0_M	s
s	s	s	s	s	s
b	0_M	s	b	b	f
c	0_M	s	b	c	f
f	s	s	f	f	f

Table 2: The nullnorm R_c on M

R_c	0_M	s	b	c	f	1_M
0_M	0_M	s	0_M	0_M	s	s
s	s	s	s	s	s	s
b	0_M	s	b	b	f	f
c	0_M	s	b	c	f	f

$\boldsymbol{f}$	s	s	f	f	f	f
$\boldsymbol{1_M}$	s	s	f	f	f	1_M

By applying Theorem 3, for an arbitrary triangular norm T_f on $[f, 1_M]$, the corresponding nullnorm R_c on M with annihilator s is shown in Table 2.

It is obvious the fact that $R_c(b, c) = b$ for the elements $b, c \in M$ from Table 2.

Let us consider any triangular norm T on $[s, 1_M]$ and any triangular conorm S on $[0_M, s]$ in order to apply Theorem 1 and Theorem 2.

By applying Theorem 1, it is obtained the fact that $R^{(T,S)}(b, c) = s$ for the elements $b, c \in M$.

By applying Theorem 2, for the elements $b, c \in M$, it is obtained the fact that $R_T^S(b, c) = S(\inf\{c, s\}, \inf\{b, s\}) = S(0_M, 0_M) = 0_M$ and $R_S^T(b, c) = T(\sup\{c, s\}, \sup\{b, s\}) = T(f, f)$ that is $R_S^T(b, c) \geq s$.

Therefore, it can be easily seen the fact that the nullnorm $R_c R$ is different from the nullnorms R_S^T, R_T^S and $R^{(T,S)}$.

In addition, Example 2 shows the fact that the nullnorm R_c on any bounded lattice having the annihilator s do not have to coincide with the nullnorms $R^{(T,S)}$, R_S^T, R_T^S even if all elements in M are comparable with f.

Remark 2. Consider the fact that $s, f \in M$ and $s, f \neq 0_M, 1_M$ and $s \in]0_M, f]$, a nullnorm R_f on $[0_M, f]$ having the annihilator s and a triangular norm T_f on $[f, 1_M]$. One can wonder whether the nullnorm R_c in Theorem 3 is always an idempotent nullnorm. In the following, it is illustrated the fact that this hypothesis need not be always true.

Take the lattice M with Hasse diagram shown in Figure 2. In the present case, it is obtained the fact that
$R_c(y, y) = R_f(\inf\{y, f\}, \inf\{y, f\}) = R_f(x, x) \leq s$.
That is, it can not be $R_c(y, y) = y$. So, R_c is not an idempotent nullnorm on M.

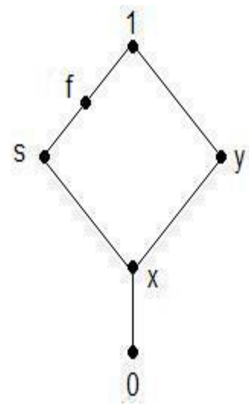

Figure 2. The lattice M

Note the fact that the bounded lattice in Figure 2 is not distributive. In that case, another natural question occurs: if M is considered as distributive lattice, does the nullnorm R_c in Theorem 3 need always be idempotent? In the following, a negative answer to this question is brought.

Take the distributive lattice M with Hasse diagram shown in Figure 3. In the present case, it is obtained the fact that $R_c(y, y) = R_f(\inf\{y, f\}, \inf\{y, f\}) = R_f(x, x) \leq f$. That is, it can not be $R_c(y, y) = y$. So, the nullnorm R_c is not idempotent on M.

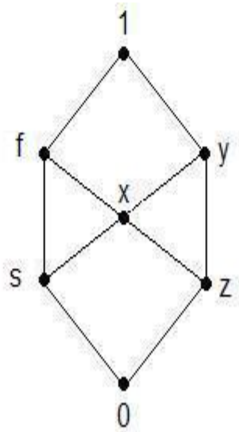

Figure 3. The lattice M

Note the fact that in the bounded lattice characterized by Figure 3, there is an element incomparable with the annihilator s. Then, one more question arises: if $I_s = \emptyset$, does the nullnorm R_c in Theorem 3 need always be idempotent? In the following, a negative answer to this question is brought.

Take the lattice M with Hasse diagram shown in Figure 4. In this case, we have the fact that

$R_c(x,x) = R_f(\inf\{x,f\}, \inf\{x,f\}) = R_f(s,s) = s$.
So, R_c is not an idempotent nullnorm on M.

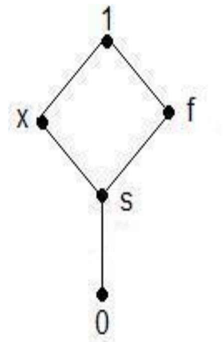

Figure 4. The lattice M

Remark 3. Consider the fact that $s, f \in M$ and $s, f \neq 0_M, 1_M$ and $s \in]0_M, f]$, a nullnorm R_f on $[0_M, f]$ having the annihilator s and a triangular norm T_f on $[f, 1_M]$. Whenever $I_f \neq \emptyset$ and we take the fact that the triangular norm T_f on $[f, 1_M]$ is the only idempotent triangular norm $T_\wedge$, in the present case, the nullnorm R_c on M given in Theorem 3 can be given by the following formula

$$R_{c(\wedge)}(p,q) = \begin{cases} R_f(p,q) & if\ (p,q) \in [0_M, f]^2, \\ \inf\{p,q\} & if\ (p,q) \in [f, 1_M]^2. \end{cases}$$

If the nullnorm R_f is idempotent on $[0_M, f]$, then the nullnorm $R_{c(\wedge)}$ is idempotent nullnorm on M having the annihilator s.

Remark 4. Consider the fact that $s, f \in M$ and $s, f \neq 0_M, 1_M$ and $s \in]0_M, f]$, a nullnorm R_f on $[0_M, f]$ having the annihilator s and a triangular norm T_f on $[f, 1_M]$. In the present case, the nullnorm R_c in Theorem 3 need not be the greatest and the smallest nullnorm on M.

For example, taking the lattice M with Hasse diagram shown in Figure 5, it is obtained that
$R_c(f,b) = R_f(\inf\{f,f\}, \inf\{b,f\}) = R_f(t, 0_M)$
$= s$ and $R^{(\vee)}(f,b) = \inf\{\sup\{f,s\}, \sup\{b,s\}\} = \inf\{f,z\} = t$ for $f, b \in M$. Since $R_c(f,b) = s \neq t = R^{(\vee)}(f,b)$, the nullnorm R_c is not the greatest nullnorm on M. In addition, since $R_c(x,y) = T_f(x,y) \geq f$ and $R^{(\wedge)}(x,y) = s$ for $x, y \in M$, the nullnorm R_c is not the smallest nullnorm on M.

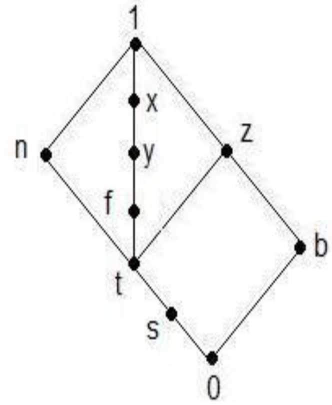

Figure 5. The lattice M

Proposition 3. Consider the fact that $s, f \in M$ and $s, f \neq 0_M, 1_M$ and $s \in]0_M, f]$, a nullnorm R_f on $[0_M, f]$ having the annihilator s and a triangular norm T_f on $[f, 1_M]$. In the present case, the nullnorm $R_c: M^2 \to M$ defined in Theorem 3 can be also written by one of the follows:

$$R_{c1}(p,q) = \begin{cases} S_f(p,q), & \text{if } (p,q) \in [0_M, s]^2 \\ T_f(p,q), & \text{if } (p,q) \in [f, 1_M[^2 \\ \inf\{p,q\}, & \text{if } p = 1_M\ or\ q = 1_M \\ s\ , & \begin{array}{l} \text{if } p \in [f, 1_M[\text{ and } q \in I_f \cap [s, 1_M[\\ \text{or } p \in I_f \cap [s, 1_M[\text{ and } q \in [f, 1_M[\\ \text{or } p \in [s, f] \text{ and } q \in I_f \cap [s, 1_M[\\ \text{or } p \in I_f \cap [s, 1_M[\text{ and } q \in [s, f] \\ \text{or } p \in [f, 1_M[\text{ and } q \in [s, f] \\ \text{or } p \in [s, f] \text{ and } q \in [f, 1_M] \\ \text{or } p, q \in I_f \cap [s, 1_M[\\ \text{or } p, q \in [s, f] \text{ or } p, q \in D_s \end{array} \\ R_f(\inf\{p,f\}, \inf\{q,f\}), & \text{otherwise} \end{cases}$$

or

$$R_{c2}(p,q) = \begin{cases} S_f(p,q), & \text{if } (p,q) \in [0_M, s]^2 \\ T_f(p,q), & \text{if } (p,q) \in [f, 1_M[^2 \\ \inf\{p,q\}, & \text{if } p = 1_M \text{ or } q = 1_M \\ s, & \text{if } p,q \in D_s \\ \inf\{p,q,f\}, & \begin{array}{l} if\ p \in [f,1_M[\text{ and } q \in I_f \cap [s,1_M[\\ \text{or } p \in I_f \cap [s,1_M[\text{ and } q \in [f,1_M[\\ \text{or } p \in [s,f] \text{ and } q \in I_f \cap [s,1_M[\\ \text{or } p \in I_f \cap [s,1_M[\text{ and } q \in [s,f] \\ \text{or } p \in [f,1_M[\text{ and } q \in [s,f] \\ \text{or } p \in [s,f] \text{ and } q \in [f,1_M] \\ \text{or } p,q \in I_f \cap [s,1_M[\\ \text{or } p,q \in [s,f] \end{array} \\ R_f(\inf\{p,f\}, \inf\{q,f\}), & \text{otherwise.} \end{cases}$$

where $S_f = R_f \upharpoonright [0_M, s]$ is a triangular conorm on $[0_M, s]$.

Proof. Since T_f is a triangular norm acting on $[f, 1]$, by using the construction methods in (Çaylı 2018) and (Ertuğrul et al. 2015), it is obtained the fact that the following triangular norms $T_1: [s, 1_M]^2 \to [s, 1_M]$ and $T_2: [s, 1_M]^2 \to [s, 1_M]$, respectively.

$$T_1(a,b) = \begin{cases} T_f(a,b) & if\ (a,b) \in [f,1_M[^2, \\ \inf\{a,b\} & if\ a = 1_M \text{ or } b = 1_M, \\ s & otherwise. \end{cases}$$

$$T_2(a,b) = \begin{cases} T_f(a,b) & if\ (a,b) \in [f,1_M[^2, \\ \inf\{a,b\} & if\ a = 1_M \text{ or } b = 1_M, \\ \inf\{a,b,f\} & otherwise. \end{cases}$$

In this case, by using the triangular norms T_1 and T_2 acting on $[s, 1_M]$, it is obtained the nullnorms $R_{c1}: M^2 \to M$ and $R_{c2} : M^2 \to M$, respectively, having the annihilator $s \in M \setminus \{0_M, 1_M\}$ by means of the construction approach in Theorem 3.

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