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Araştırma Makalesi / Research Article

On The Area Of A Triangle In $\mathrm{IR}^2_{\mathrm{M}}$

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Abstract

There are known to be many methods for calculating the area of triangle. It is far easier to determine the area as long as the length of all the three sides of the triangle are known. What appears to be essential here is the way in which the lengths are to be measured. The present study aims to present three methods for calculating the area of a triangle by achieving the measuring process via the maximum metric d_M in preference to the usual Euclidean metric d_E.

$\mathrm{IR}^2_{\mathrm{M}}$ de Bir Üçgenin Alanı Üzerine

Anahtar kelimeler
Uzaklık Fonksiyonu; Maksimum Metrik; Heron Formülü; İsometri Grup.

Özet

Bir üçgenin alanını hesaplamak için birçok yöntem bilinmektedir. Üçgenin kenar uzunlukları bilindiğinde alanı hesaplamak oldukça kolaydır. Burada ki en temel problem uzunlukların nasıl ölçüldüğüdür. Bu makalede uzunluklar iyi bilinen Euclidean metrik d_E yerine d_M metriği kullanılarak, bir üçgenin alanını hesaplamakla ilgili olarak üç yöntem verilecektir.

1. Introduction

The maximum geometry has applications in the real world and it can be considered good model for the world. For example, urban planning at the macro dimensions can be used for approximate calculations. To measure distance the between two points $A=(x_1,y_1)$, $B=(x_2,y_2)$ in the analytical plane IR^2, it is not need to use the well-known Euclidean metric

$$d_E(A,B)=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}.$$

We can use the maximum metric

$$d_M(A,B)=|x_1-x_2|+|y_1-y_2|$$

instead of Euclidean metric. Linear structure except distance function in the $\mathrm{IR}^2_{\mathrm{M}}$ is the same as Euclidean analytical plane. Since analytical plane is furnished by a different distance function, well known results about distance concept in Euclidean geometry can be changed. It is important to work on concepts related to the distance in geometric studies, because change of metric can reveals interesting results. For example, areas of triangles having congruent side lengths may be different (Figure 1). Let us consider the unit circles C_1 and C_2 in the $\mathrm{IR}^2_{\mathrm{M}}$ and A, B be center $M-$ circles C_1 and C_2, respectively. Also C and D are two points in $C_1\cap C_2$.

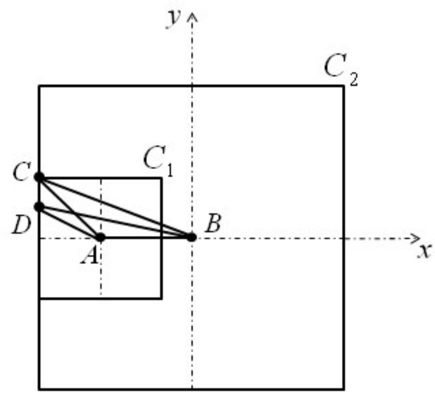


Figure 1. The Triangles Having Different Area

If one can calculate the areas of triangle ABC and ABD using Euclidean area notion, then

$$Area(\Delta ABC)\neq Area(\Delta ABD)$$

although $d_M(A,C)=d_M(A,D)$ and $d_M(B,C)=d_M(B,D)$. Therefore how to calculate the area of triangle in the $\mathrm{IR}^2_{\mathrm{M}}$ is an important question. To compute the area of a triangle in the $\mathrm{IR}^2_{\mathrm{M}}$, we give three different formulas depend on some parameters in the rest of the article.

For the sake of simple, the maximum plane will be denoted by $\mathrm{IR}^2_{\mathrm{M}}$, and a triangle with vertices A, B and C is denoted ΔABC in the rest of the article. Also maximum lengths of sides AB, AC and BC of triangle ABC will be denoted by c_M, b_M and a_M respectively. That is $c_M = d_M(A,B)$, $b_M = d_M(A,C)$ and $a_M = d_M(B,C)$. Similarly, Euclidean lengths of sides AB, AC and BC of the triangle ABC will be denoted by $c = d_E(A,B)$, $b = d_E(A,C)$ and $a = d_E(B,C)$.

2. Area of a triangle in $\mathbf{IR}^2_{\mathbf{M}}$

It is easy to find the area of a triangle, in the Euclidean plane; area of a triangle is half of the base times the height, where the base and height of a triangle must be perpendicular to each other. Then, area of ΔABC equal to

$$Area(\Delta ABC) = ah/2$$

where $h = d_E(A,H) = d_E(A,BC)$.

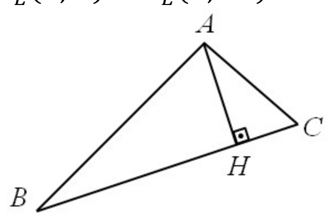


Figure 2. The Base and Height of a Triangle

In this section, we will give following propositions without proofs. Therefore, calculation process of the area formulas in terms of the metric d_M will be shortened by these propositions.

Proposition 2.1. Given any distinct two points A and B in the $M-$ plane. Let slope of the line l through A and B be m, then,

$$d_E(A,B) = \rho(m) d_M(A,B)$$

where $\rho(m) = \sqrt{1+m^2}/max\{1,|m|\}$. If $m = 0$ or $m \to \infty$, then $d_E(A,B) = d_M(A,B)$.

Following proposition is a result of Proposition 2.1:

Proposition 2.2. For any $m \in IR - \{0\}$, then

$$\rho(m) = \rho(-m) = \rho(1/m) = \rho(-1/m).$$

We immediately say by Proposition 2.1 and Proposition 2.2 that the maximum distance between two points is invariant under all translations, rotations of $\pi/2$, π and $3\pi/2$ radians around a point and the reflections about the lines parallel to $x = 0$, $y = 0$, $y = x$ or $y = -x$ (Salihova, 2006). Also isometry group of maximum plane is the semi-product of D_4 and $T(2)$ where D_4 is symmetry group of square and D_4 is set of all translation of Euclidean plane.

The following theorem gives an $M-$ version circles of the well-known Euclidean area formula of a triangle:

Theorem 2.1. For any triangle ABC in $\mathrm{IR}^2_{\mathrm{M}}$, let foot of perpendicular line from A down to the base BC of the triangle be labeled H.

i) If side BC of ΔABC is parallel to x or $y-$ axis, then

$$Area(\Delta ABC) = \frac{a_M h_M}{2}.$$

ii) If side BC of ΔABC is not parallel to x or $y-$ axis, then

$$Area(\Delta ABC) = [\rho(m)]^2 \frac{a_M h_M}{2}$$

where $\rho(m) = \sqrt{1+m^2}/max\{1,|m|\}$ (Figure 2).

Proof : $Area(\Delta ABC) = ah/2$ such that $h = d_E(A,H) = d_E(A,BC)$.

i) If side BC of ΔABC is parallel to x or $y-$ axis, then clearly $a_M = a$ and $h_M = d_M(A,H) = h$. So,

$$Area(\Delta ABC) = \frac{a_M h_M}{2}.$$

ii) If side BC of ΔABC is not parallel to x or $y-$ axis, and if the slope of the line BC is m, then the slope of the line AH is $-1/m$. Consequently the equations $a = \rho(m) a_M$ and $h = \rho(m) h_M$ are obtained by Proposition 2.1 and Proposition 2.2. Therefore

$$Area(\Delta ABC) = [\rho(m)]^2 \frac{a_M h_M}{2}.$$

In the following section, we will give another area formula depending on the parameter m, using

d_M −distance from a point to a line in $\mathrm{IR}^2_{\mathrm{M}}$.

In $\mathrm{IR}^2_{\mathrm{M}}$, d_M −distance from a point P to a line l is characterized in that

$$d_M(P,l) = \min_{Q \in l}\{d_M(P,Q)\}.$$

This characterization is defined similarly in the Euclidean plane. That is, let $P = (x_0, y_0)$ be any point and $l : ax + by + c = 0$ be any line, then d_E −distance from the point P to the line l is

$$d_M(P,l) = \frac{|ax_0+by_0+c|}{\sqrt{a^2+b^2}}.$$

Following proposition gives d_M −distance from a point P to a line l.

Proposition 2.3. $\mathrm{IR}^2_{\mathrm{M}}$, d_M −distance from a point $P = (x_0, y_0)$ to a line l with the equation $ax + by + c = 0$ is

$$d_M(P,l) = |ax_0 + by_0 + c|/max\{|a+b|, |a-b|\}$$

Proof : Think of slowly inflating $M-$ circle with center P and radius 0 with it just touches l. Its radius at that moment is $d_M(P,l)$. This means that the line l becomes tangent to the $M-$circle. Therefore, minimum d_M −distance from the point P to the line l can be easily calculated (Krause, 1987). $M-$circle touches l at one vertex or one edge. The points where the $M-$circle touches the line l must be on the lines with slopes ± 1 through P. If tangent points are calculated, then $P_1 = \left(\frac{bx_0-by_0-c}{a+b}, \frac{-ax_0+ay_0-c}{a+b}\right)$ and $P_2 = \left(\frac{-bx_0-by_0-c}{a-b}, \frac{ax_0+ay_0+c}{a-b}\right)$ are obtained where P_1 is the common points of line l and $-x + y + x_0 - y_0 = 0$, similarly P_2 is the common points of line l and $x + y - x_0 - y_0 = 0$ (Figure 3).

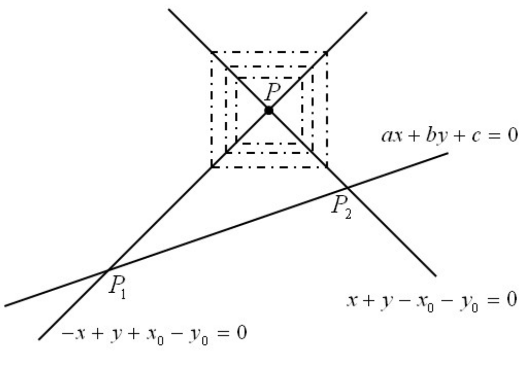


Figure 3. Inflating $M-$Circle with Center P

Consequently,

$$d_M(P,l) = min\{d_M(P,P_i) : i = 1,2\}$$

$$d_M(P,l) = min\left\{\frac{|ax_0+by_0+c|}{|a+b|}, \frac{|ax_0+by_0+c|}{|a-b|}\right\}.$$

$$= \frac{|ax_0+by_0+c|}{max\{|a+b|,|a-b|\}}$$

The next proposition gives an explanation about relation between d_E −distance and d_M −distance from any point P to any line l. Then, this relation will be used in the following theorem which contains another area formula depending on the parameter m which is slope of line l.

Proposition 2.4. Let $P = (x_0, y_0)$ be point, and l with slope m be a line in $\mathrm{IR}^2_{\mathrm{M}}$. Then

$$d_E(P,l) = \tau(m) d_M(P,l)$$

where $\tau(m) = max\{|1+m|, |1-m|\}/\sqrt{1+m^2}$ and $m \in IR - \{0\}$.

Proof : The d_E and d_M − distances from the point $P = (x_0, y_0)$ be point to the line l with the equation $ax + by + c = 0$ for $m \neq 0$, respectively $d_E(P,l) = \frac{|ax_0+by_0+c|}{|b|\sqrt{1+m^2}}$ and $d_M(P,l) = \frac{|ax_0+by_0+c|}{|b| max\{|1+m|,|1-m|\}}$. So, $d_E(P,l) = \tau(m) d_M(P,l)$ where $\tau(m) = max\{|1+m|, |1-m|\}/\sqrt{1+m^2}$.

If $m \to \infty$, then $b = 0$ and $a \neq 0$. Therefore, we get $d_E(P,l) = \frac{|ax_0+c|}{|a|}$ and $d_M(P,l) = \frac{|ax_0+c|}{|a|}$. So $d_E(P,l) = d_M(P,l)$. Similary, if $m = 0$ then $d_E(P,l) = d_M(P,l)$.

Theorem 2.2. Given ΔABC in $\mathrm{IR}^2_{\mathrm{M}}$. Let m be the slope of the line BC, and $a = d_M(B,C)$ and $h = d_M(A,H)$.

i) If side BC of ΔABC is parallel to x or $y-$ axis, then

$$Area(\Delta ABC) = \frac{a_M h_M}{2}.$$

ii) If side BC of ΔABC is not parallel to x or $y-$ axis, then

$$Area(\Delta ABC) = \sigma(m)\frac{a_M h_M}{2}$$

where $\sigma(m) = \frac{max\{|1+m|,|1-m|\}}{max\{1,|m|\}}$.

Proof : $Area(\Delta ABC) = ah/2$ such that $h = d_E(A,H) = d_E(A,BC)$.

i) The proof can be given by similar way in Theorem 2.1.

ii) If side BC of ΔABC is not parallel to x or $y-$ axis, and the slope the line BC is m, then equations $a = \rho(m)a_M$ and $h = \tau(m)h_M$ are obtained by Proposition 2.1 and Proposition 2.2. Therefore

$$Area(\Delta ABC) = \rho(m)\tau(m)\frac{a_M h_M}{2}$$

Since $\rho(m)\tau(m) = \sigma(m)$, we get

$$Area(\Delta ABC) = \sigma(m)\frac{a_M h_M}{2}.$$

3. Heron' s Formula in $\mathbf{IR}^2_{\mathbf{M}}$

In geometry, Heron's formula (sometimes called Hero's formula) is called after Hero of Alexandria. A method has been know for nearly 2000 years for calculating the area of a triangle when you know the lengths of all three sides. Heron' s formula gives the area of ΔABC as

$$Area(\Delta ABC) = \sqrt{p(p-a)(p-b)(p-c)}$$

where p is the semiperimeter of the triangle; that is, $p = \frac{a+b+c}{2}$.

In this section, our next step is to give that some definitions. To do this it will be useful to give $M-$ version of the Heron' s formula depend on semiperimeter, lengths of sides of the triangle and the new parameter. These definitions are given in (Colakoglu and Kaya, 2002), (Gelisgen and Kaya, 2009), (Salihova, 2006).

Definition 3.1. Let ΔABC be any triangle in $\mathrm{IR}^2_{\mathrm{M}}$. Clearly, there exists a pair of lines passing through every vertex of the triangle, each of which is parallel to a coordinate axis. A line l is called a base line of ΔABC iff

1) l passes through a vertex;
2) l is parallel to a coordinate axis;
3) l intersects the oppsite side (as a line segment) of vertex in condition 1.

Clearly, at least one of vertices of the triangle always has one or two base lines. Such a vertex of the triangle is called a basic vertex. A base segment is a line segment on a base line, which is bounded by a basic vertex and its opposite side.

Definition 3.2. A line with slope m is called a steep line, a gradual line and a separator if $|m| > 1$, $|m| < 1$ and $|m| = 1$, respectively. In particularly, a gradual line is called horizontal if it is parallel to $x-$axis, and a steep line is called vertical if it is parallel to $y-$axis.

A vertex of a triangle ΔABC can be taken at origin since all translations of the analytical plane are isometries of $\mathrm{IR}^2_{\mathrm{M}}$. So we can take the vertex C such that it is at origin in the rest of the article. Also we will label vertices of the triangle in counterclockwise order. We are now ready to give M-version of the Heron' s formula in the following theorems.

Note that the following theorem gives $M-$ version of the Heron' s formula when one side of a ΔABC is parallel to one of the coordinate axes. If two sides of a ΔABC are parallel to the coordinate axes, then area of ΔABC equals to half of product maximum distance of perpendicular sides.

Theorem 3.1. Given ΔABC such that the side AC of ΔABC is parallel to $x-$axis in $\mathrm{IR}^2_{\mathrm{M}}$. Let foot of perdendicular line from B down to the line AC be labeled. Then $M-$ version of the Heron' s formula is

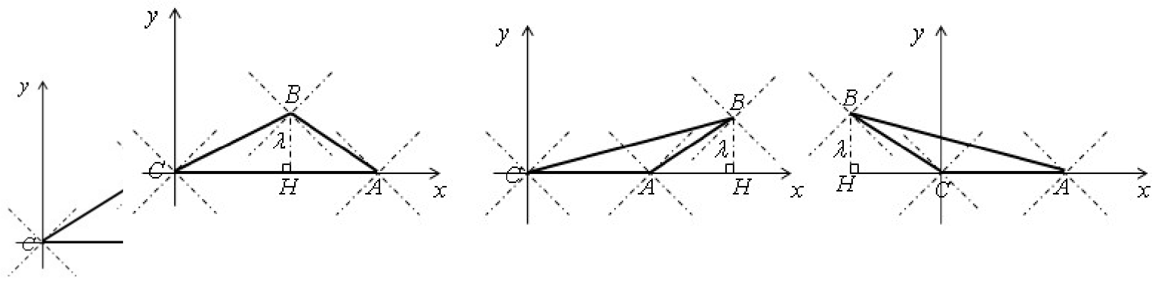
$$Area(\Delta ABC) = \frac{1}{2}\lambda(2p - a_M - c_M)$$

where $\lambda = d_M(B,H)$ and p_M is the $M-$ version of semiperimeter of the triangle; that is $p_M = (a_M + b_{M+}c_M)/2$.

Proof : Given ΔABC in $\mathrm{IR}^2_{\mathrm{M}}$. Let side AC of ΔABC be parallel to $x-$axis. Then the triangle ΔABC can be classified as four groups according to slopes of the sides of triangles:

i) AB and BC sides of the triangle lie on gradual (steep) lines (Figure 5).

ii) AB side of the triangle lie on gradual (steep) lines, BC side lies on a steep (gradual) line (Figure 4).



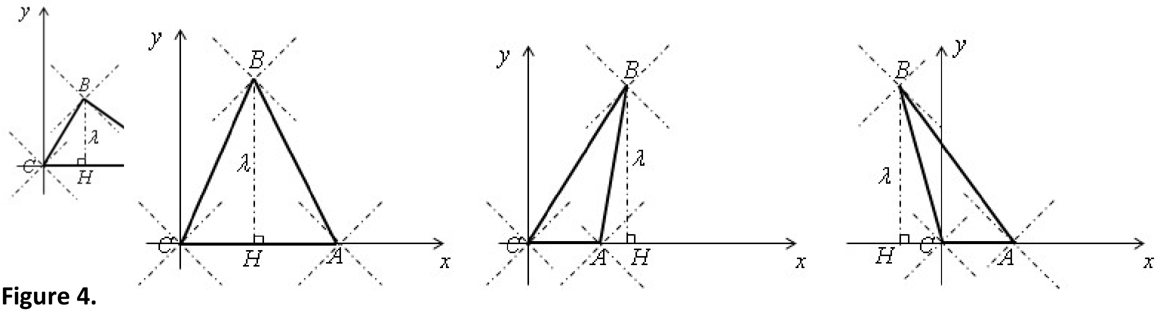


Figure 4.
The Triangles Are Parallel to $x-$ Axis.

Figure 3.2

Figure 5. The Triangles Are Parallel to $x-$ Axis.

Case i) Let sides AB and BC of the triangle be on gradual (steep) lines. Then

$$\mathrm{Area}(\Delta\mathrm{ABC}) = \frac{1}{2} d_E(\mathrm{B},\mathrm{H}).\, d_E(\mathrm{C},\mathrm{A})$$

$$= \tfrac{1}{2} d_M(\mathrm{B},\mathrm{H}).\, d_M(\mathrm{C},\mathrm{A})$$

$$= \tfrac{1}{2} d_M(\mathrm{B},\mathrm{H}).\, c_M$$

$$= \tfrac{1}{2} d_M(\mathrm{B},\mathrm{H}).\, (2p - a_M - c_M)$$

Note that Figure 4 and Figure 5 represent all the triangles that side AC of $\Delta\mathrm{ABC}$ be parallel to $x-$ axis. If one side of $\Delta\mathrm{ABC}$ is parallel to $x-$axis, then the role of vertices A and B of the triangle must be replace with each other. Consequently,

$$\mathrm{Area}(\Delta\mathrm{ABC}) = \frac{1}{2} d_M(\mathrm{A},\mathrm{H}).\, (2p - b_M - c_M)$$

is obtained.

Case ii) The proof can be immediately given as in Case i.

Note that we give $M-$version of the Heron' s formula in the following theorem when any sides of $\Delta\mathrm{ABC}$ is not parallel to any one of the coordinate axes.

Theorem 3.2. Given $\Delta\mathrm{ABC}$ such that C is a basic vertex in $\mathrm{IR}^2_{\mathrm{M}}$. Let foot of base line from the basic vertex C down to the side AB of the triangle be labeled by D. Then the $M-$version of the Heron' s area formula is equals to

$$\mathrm{Area}(\Delta\mathrm{ABC}) \;=\; \tfrac{1}{2}\lambda(2p - a_M - b_M)$$

where p_M is the $M-$version of semiperimeter of the triangle; that is, $p_M = (a_M + b_M + c_M)/2$ and $\lambda = d_M(\mathrm{D},\mathrm{C})$.

Proof : Given $\Delta\mathrm{ABC}$ in $\mathrm{IR}^2_{\mathrm{M}}$. To show Heron' s formula for the $\Delta\mathrm{ABC}$ whose sides are not parallel to one of the coordinate axes, we shall consider following cases;

i) Sides AC and BC of the triangle are not gradual and base line CD is horizontal, or sides AC and BC of the triangle are not steep and base line CD is vertical (Figure 6-a-b);

ii) Sides AC and BC of the triangle are not steep and base line CD is horizontal, or sides AC and BC of the triangle are not gradual and base line CD is vertical (Figure 6-c);

iii) Side AC of the triangle are not gradual, side BC is not steep and base line CD is horizontal, or side AC of the triangle is not steep, side BC is not gradual and base line CD is vertical (Figure 6-d-e);

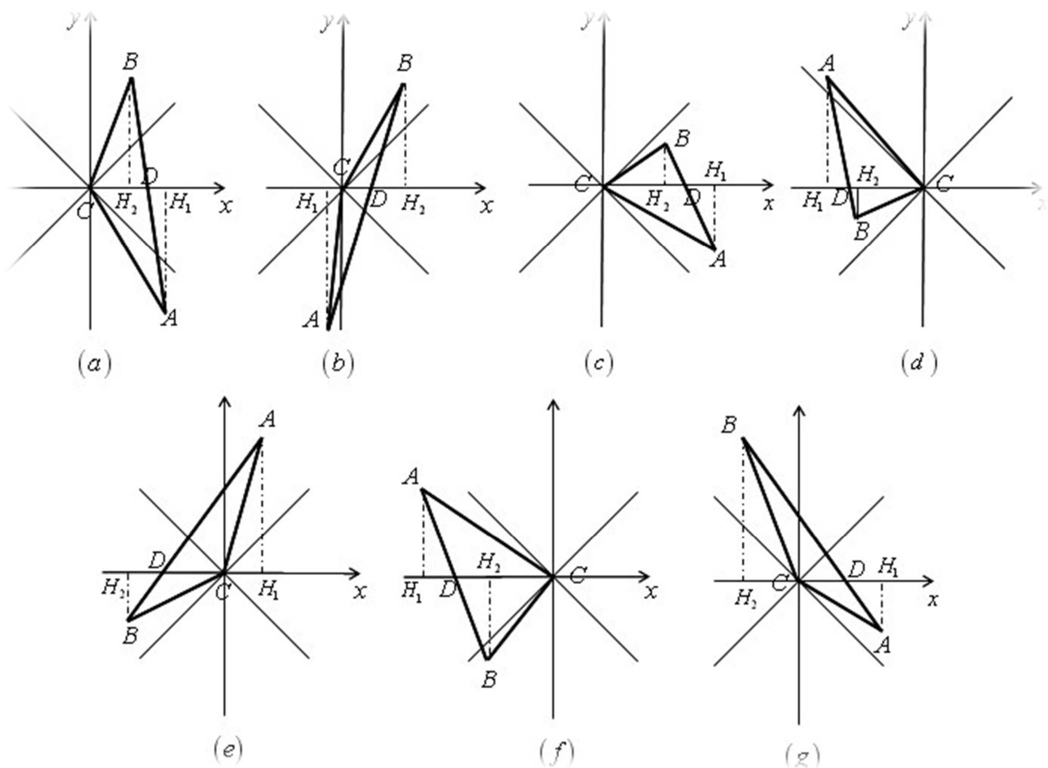


Figure 6. The Triangles in m-Plane

iv) Side AC of the triangle are not steep, side BC is not gradual and base line CD is horizontal, or side AC of the triangle is not gradual, side BC is not steep and base line CD is vertical (Figure 6-f-g);

Case i) Let sides AC and BC sides of the triangle be not gradual and base line CD be horizontal (Figure 6-a-b). Then

$$\mathrm{Area}(\Delta\mathrm{ABC}) = \mathrm{Area}(\Delta\mathrm{ADC}) + \mathrm{Area}(\Delta\mathrm{DCB})$$

$$\mathrm{Area}(\Delta\mathrm{ABC}) = \frac{1}{2} d_E(\mathrm{A},\mathrm{CD}).\, d_E(\mathrm{C},\mathrm{D}) + \frac{1}{2} d_E(\mathrm{B},\mathrm{CD}).\, d_E(\mathrm{C},\mathrm{D})$$

$$\mathrm{Area}(\Delta\mathrm{ABC}) = \frac{1}{2} d_M(\mathrm{A},\mathrm{CD}).\, d_M(\mathrm{C},\mathrm{D}) + \frac{1}{2} d_M(\mathrm{B},\mathrm{CD}).\, d_M(\mathrm{C},\mathrm{D})$$

$$\text{Area}(\Delta \text{ABC}) = \frac{1}{2} d_M(\text{C}, \text{D}) c_M$$

$$\text{Area}(\Delta \text{ABC}) = \frac{1}{2} d_M(\text{C}, \text{D})[d_M(\text{A}, \text{CD}) + d_M(\text{B}, \text{CD})]$$

$$\text{Area}(\Delta \text{ABC}) = \frac{1}{2} d_M(\text{C}, \text{D}) c_M$$

$$\text{Area}(\Delta \text{ABC}) = \frac{1}{2} d_M(\text{C}, \text{D}) c_M$$

$$\text{Area}(\Delta \text{ABC}) = \frac{1}{2} d_M(\text{C}, \text{D})(2p - a_M - c_M)$$

Note that Figure 6 represent all the triangles providing the classification which is mentioned the beginning of the proof. The proof of the other cases can be given by similar way. Therefore, we have shown that $M -$version of the Heron's formula is equals to

$$\text{Area}(\Delta \text{ABC}) \;=\; \frac{1}{2} \lambda(2p - a_M - b_M)$$

where $\lambda = d_M(\text{D}, \text{C})$ and $p = (a_M + b_M + c_M)/2$.

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