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Araştırma Makalesi / Research Article

İzotropik 3-Uzayda Yüzeyler Üzerine Sınıflandırma Sonuçları

Muhittin Evren Aydın
Fırat Üniversitesi, Fen Fakültesi, Matematik Bölümü, Elazığ
e-mail: meaydin@firat.edu.tr

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Anahtar kelimeler
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Özet

İzotropik 3-uzay $\mathbb{I}^3$ Cayley-Klein uzaylarından biridir ve Öklidyen uzayda standart Öklidyen uzaklık ile izotropik uzaklığın değişiminden elde edilir. Bu çalışmada, $\mathbb{I}^3$ uzayında, sabit relatif (izotropik Gauss) ve sabit izotropik ortalama eğrilikli yüzeyler üzerine çeşitli sınıflandırmalar ifade edilmiştir. Özel olarak, $\mathbb{I}^3$ uzayında sabit eğrilikli helikoidal yüzeyler sınıflandırılıp, bu yüzeyler üzerinde bazı özel eğriler analiz edilmiştir.

Classification Results on Surfaces in The Isotropic 3-Space

Keywords
Isotropic space; Helicoidal surface; Isotropic mean curvature; Relative curvature; Geodesics; Asymptotic curve.

Abstract

The isotropic 3-space $\mathbb{I}^3$ which is one of the Cayley-Klein spaces is obtained from the Euclidean space by substituting the usual Euclidean distance with the isotropic distance. In the present paper, we give several classifications on the surfaces in $\mathbb{I}^3$ with constant relative curvature (analogue of the Gaussian curvature) and constant isotropic mean curvature. In particular, we classify the helicoidal surfaces in $\mathbb{I}^3$ with constant curvature and analyze some special curves on these.

1. Introduction

Differential geometry of isotropic spaces have been introduced by Strubecker (1942), Sachs (1978, 1990a, 1990b), Palman (1979) and others. Especially the reader can find a well bibliography for isotropic planes and isotropic 3-spaces in Sachs (1990a, 1990b).

The isotropic 3-space I^3 is a Cayley-Klein space defined from a 3-dimensional projective space $P(\mathbb{R}^3)$ with the absolute figure which is an ordered triple (ω, f_1, f_2), where ω is a plane in $P(\mathbb{R}^3)$ and f_1, f_2 are two complex-conjugate straight lines in ω, see (Milin Sipus, 2014). The homogeneous coordinates in $P(\mathbb{R}^3)$ are introduced in such a way that the *absolute plane* ω is given by $X_0 = 0$ and the *absolute lines* f_1, f_2 by

$$X_0 = X_1 + iX_2 = 0, \; X_0 = X_1 - iX_2 = 0.$$

The intersection point $F(0:0:0:1)$ of these two lines is called the *absolute point*. The group of motions of I^3 is a six-parameter group given in the affine coordinates $x_1 = \frac{X_1}{X_0}$, $x_2 = \frac{X_2}{X_0}$, $x_3 = \frac{X_3}{X_0}$ by

$$(x_1, x_2, x_3) \mapsto (x_1', x_2', x_3')$$

$$(x_1', x_2', x_3') : \begin{cases} x_1' = a + x_1 \cos\phi - x_2 \sin\phi, \\ x_2' = b + x_1 \sin\phi + x_2 \cos\phi, \\ x_3' = c + dx_1 + ex_2 + x_3, \end{cases} \quad (1.1)$$

where $a, b, c, d, e, \phi \in \mathbb{R}$.

Such affine transformations are called *isotropic congruence transformations* or *i-motions.* It easily seen from (1.1) that i-motions are indeed composed by an Euclidean motion in the x_1x_2-plane (i.e. translation and rotation) and an affine shear transformation in x_3-direction.

Consider the points $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$. The projection in x_3-direction onto $\mathbb{R}^2$, $(x_1, x_2, x_3) \mapsto (x_1, x_2, 0)$, is called the *top view*. The *isotropic distance*, so-called *i-distance* of two points $\mathbf{x}$ and $\mathbf{y}$ is defined as the Euclidean distance of their top views, i.e.,

$$\|\mathbf{x}-\mathbf{y}\|_i = \sqrt{\sum_{j=1}^{2}\left(y_j - x_j\right)^2}. \tag{1.2}$$

The i-metric is degenerate along the lines in x_3- direction, and such lines are called *isotropic* lines. The plane containing an isotropic line is called an *isotropic plane.*

Let M^2 be a surface immersed in $\mathbb{I}^3$. M^2 is called *admissible* if it has no isotropic tangent planes. We restrict our framework to admissible regular surfaces. For such a surface M^2 the coefficients a_{11}, a_{12}, a_{22} of its first fundamental form are calculated with respect to the induced metric.

The normal vector field of M^2 is always the isotropic vector $(0,0,1)$ since it is perpendicular to all tangent vectors to M^2.

The coefficients b_{11}, b_{12}, b_{22} of the second fundamental form of M^2 are calculated with respect to the normal vector field of M^2. For details, see Sachs (1990b), p. 155.

The *relative curvature* (so called *isotropic Gaussian curvature*) and *isotropic mean curvature* are respectively defined by

$$K = \frac{\det(b_{ij})}{\det(a_{ij})}, \quad H = \frac{a_{11}b_{22} - 2a_{12}b_{12} + a_{22}b_{11}}{2\det(a_{ij})}.$$

The surface M^2 is said to be *isotropic flat* (resp. *isotropic minimal*) if K (resp. H) vanishes.

The curves and surfaces in the isotropic spaces have been studied by Kamenarovic (1982, 1994), Pavkovic (1980) and Divjak and Milin Sipus (2008), Milin Sipus and Divjak (1998).

Most recently, Milin Sipus (2014) classified the translation surfaces of constant curvature generated by two planar curves in $\mathbb{I}^3$. And then some classifications for the ones generated by a space curve and a planar curve with constant curvature were obtained in (Aydin, 2015).

Aydin and Mihai (2016) established a method to calculate the second fundamental form of the surfaces of codimension 2 in the isotropic 4-space $\mathbb{I}^4$ and classified some surfaces in $\mathbb{I}^4$ with vanishing curvatures.

In this paper, the helicoidal surfaces in $\mathbb{I}^3$ with constant isotropic mean and constant relative curvature are classified. Further some special curves on such surfaces are characterized.

2. Classifications of surfaces in isotropic spaces

This section is devoted to recall the classification results on hypersurfaces (also surfaces of codimension 2) in the isotropic $(n+1)-$space $\mathbb{I}^{n+1}$ $(n \geq 2)$ into seperate subsections, such as the translation hypersurfaces, the homothetical hypersurfaces (so-called factorable surfaces), Aminov surfaces, the spherical product surfaces.

2.1. Translation hypersurfaces in $\mathbb{I}^{n+1}$

The present author introduced the translation surfaces in $\mathbb{I}^3$ generated by a space curve and a planar curve as follows (for details, see Aydin (2015))

$$\mathbf{r}(u_1,u_2) \mapsto \left(f_1(u_1), f_2(u_1)+g_2(u_2), f_3(u_1)+g_3(u_2)\right), \tag{2.1}$$

and classified the ones with constant curvature by the following theorems:

Theorem 2.1. (Aydin, 2015) *Let M^2 be a translation surface given by* (2.1) *in* $\mathbb{I}^3$ *with constant relative curvature K_0. Then it is either a generalized cylinder, i.e. $K_0 = 0$, or parametrized by one of the following*

(i)

$$\mathbf{r}(u_1,u_2) = (f_1, \beta_1 f_1 + g_2, \alpha_1(f_1)^2 + \tfrac{K_0}{\alpha_1}(g_2)^2 + \beta_2 f_1 + \beta_3 g_2);$$

(ii)

$$\mathbf{r}(u_1,u_2) = (f_1, \alpha_2(f_1)^2 + \beta_4 f_1 + g_2, \beta_5(f_1)^2 + \tfrac{1}{K_0\alpha_3}(-2K_0 g_2)^{3/2} + \beta_6 f_1 + \alpha_4 g_2),$$

where α_i are nonzero constants and β_j some constants for $1 \leq i \leq 4$ and $1 \leq j \leq 6$.

Theorem 2.2. (Aydin, 2015) *Let M^2 be a translation surface given by (2.1) in $\mathbb{I}^3$ with constant isotropic mean curvature H_0. Then it is determined by one of the following expressions*

(i)

$$\mathbf{r}(u_1,u_2) = \left(f_1, f_2 + g_2, H_0 f_1^2 + \beta_1 f_2 + \beta_2 f_1 + \beta_3 g_2\right),$$

(ii)

$$\mathbf{r}(u_1,u_2) = (f_1, \beta_4 f_1 + g_2, (H_0 - \alpha_1)(f_1)^2 + \alpha_2(g_2)^2 + \beta_5 f_1 + \beta_6 g_2);$$

(iii)

$$\mathbf{r}(u_1,u_2)=\left(f_1,-\tfrac{1}{\alpha_3}\ln\left|\cos(\alpha_3 f_1)\right|+g_2,H_0 f_1^2+\beta_7 f_2+\tfrac{1}{\alpha_3^2}\exp(\alpha_3 g_2)+\beta_8 f_1+\beta_9 g_2\right),$$

where α_i *are nonzero constants and* β_i *some constants* $1\le i\le 3$ and $1\le j\le 9$.

Remark 2.3. *Isotropic minimal translation surfaces can also be classified by Theorem 2.2 as taking* $H_0=0$ *in the statements (i)-(iii) of the theorem.*

A *translation hypersurface* (M^n,F) in $\mathbb{I}^{n+1}$ is parametrized by

$$X:\mathbb{R}^n\to\mathbb{I}^{n+1},\ \mathbf{x}\mapsto(\mathbf{x},F(\mathbf{x})),$$

$$F(\mathbf{x}):=\sum_{j=1}^{n}f_j(x_j),\ \mathbf{x}\in\mathbb{R}^n,$$

where f_j are smooth functions of one variable for all $j\in\{1,\dots,n\}$, (Aydin and Ogrenmis, 2016).

For more details of $\mathbb{I}^{n+1}$, see (Chen et al., 2014), (Sachs, 1978) and (Milin Sipus and Divjak, 1998).

Some classifications were obtained for such hypersurfaces in $\mathbb{I}^{n+1}$ via the following results:

Theorem 2.4. (Aydin and Ogrenmis, 2016) *Let* (M^n,F) *be a translation hypersurface in* $\mathbb{I}^{n+1}$ *with nonzero constant relative curvature* K_0. *Then it has of the form*

$$X(\mathbf{x})=\left(\mathbf{x},\sum_{j=1}^{n}\alpha_j x_j^2+\beta_j x_j+\varepsilon\right),$$

where $\mathbf{x}\in\mathbb{R}^n$, α_j are nonzero constants and β_j,ε *some constants for all* $j\in\{1,\dots,n\}$ *such that* $\prod_{j=1}^{n}\alpha_j=\frac{1}{2^n}K_0$.

In particular, if (M^n,F) *is isotropic flat in* $\mathbb{I}^{n+1}$, *then it is congruent to a cylinder from Euclidean perspective.*

Theorem 2.5. (Aydin and Ogrenmis, 2016) *Let* (M^n,F) *be a translation hypersurface in* $\mathbb{I}^{n+1}$ *with constant isotropic mean curvature* H_0. *Then it has of the form*

$$X(\mathbf{x})=\left(\mathbf{x},\sum_{j=1}^{n}\alpha_j x_j^2+\beta_j x_j+\varepsilon\right),$$

where $\mathbf{x}\in\mathbb{R}^n$ *and* $\alpha_j,\beta_j,\varepsilon$ *are some constants for all* $j\in\{1,\dots,n\}$ *such that* $\sum_{j=n}^{n}\alpha_j=\frac{n}{2}H_0$.

Remark 2.6. *Isotropic minimal translation hypersurfaces in* $\mathbb{I}^{n+1}$ *are also classified by Theorem 2.5 as taking* $H_0=0$.

2.2. Homothetical hypersurfaces in $\mathbb{I}^{n+1}$

Aydin and Ogrenmis (2016) defined the homothetical hypersurfaces in $\mathbb{I}^{n+1}$ as follows: A hypersurface M^n of $\mathbb{I}^{n+1}$ is called a *homothetical hypersurface* (M^n,H) if it is the graph of a function of the form:

$$H(x_1,\dots,x_n):=h_1(x_1)\cdot\ldots\cdot h_n(x_n),$$

where $h_1,\dots,h_n$ are smooth non-constant functions of one real variable.

Next results classify the homothetical hypersurfaces in $\mathbb{I}^{n+1}$ with constant isotropic mean and relative curvature.

Theorem 2.7. (Aydin and Ogrenmis, 2016) *Let* (M^n,H) *be a homothetical hypersurface in* $\mathbb{I}^{n+1}$ *with constant isotropic mean curvature* H_0. *Then it is isotropic minimal, i.e.* $H_0=0$ *and has one of the following forms*

(i)

$$X(\mathbf{x})=\left(\mathbf{x},\prod_{j=1}^{n}\{\gamma_j x_j+\varepsilon_j\}\right),$$

where $\mathbf{x}\in\mathbb{R}^n$ *and* γ_j,ε_j *some constants;*

(ii)

$$X(\mathbf{x})=\left(\mathbf{x},\prod_{j=1}^{n}\left\{\gamma_j\exp\left(\sqrt{\alpha_j}x_j\right)+\beta_j\exp\left(-\sqrt{\alpha_j}x_j\right)\right\}\right),$$

for $\mathbf{x}\in\mathbb{R}^n$ *and nonzero constants* $\alpha_j,\beta_j,\gamma_j$, $j\in\{1,\dots,n\}$ *such that* $\sum_{j=1}^{n}\alpha_j=0.$

Theorem 2.8. (Aydin and Ogrenmis, 2016) *Let* (M^n,H) *be an isotropic flat homothetical hypersurface in* $\mathbb{I}^{n+1}$. *Then it has one of the following forms:*

(i) $$X(\mathbf{x})=\left(\mathbf{x},\gamma\exp(\alpha_1x_1+\alpha_2x_2)\prod_{j=3}^{n}h_j(x_j)\right)$$

for nonzero constants γ,α_1,α_2;

(ii) $$X(\mathbf{x}) = \left(\mathbf{x}, \gamma \prod_{j=1}^{n} \left(x_j + \beta_j\right)^{\alpha_j}\right),$$

where $\mathbf{x} \in \mathbb{R}^n$, γ, α_j *are nonzero constants and* β_j *some constants,* $j \in \{1,\dots,n\}$ *such that* $\sum_{i=1}^{n} \alpha_i = 1$.

2.3. Spherical product surfaces and Aminov surfaces in $\mathbb{I}^4$

The present author and I. Mihai (see Aydin and Mihai (2016)) established a method to calculate the second fundamental form of the surfaces of codimension 2 in the isotropic 4-space $\mathbb{I}^4$. Then ones classified the Aminov surfaces, given by

$$\mathbf{r} : I \times [0, 2\pi) \to \mathbb{I}^4,$$
$$(u,v) \mapsto \mathbf{r}(u,v) := (u, v, r(u)\cos v, r(u)\sin v),$$

with vanishing curvature as follows:

Theorem 2.9. (Aydin and Mihai, 2016) *The isotropic flat Aminov surfaces in* $\mathbb{I}^4$ *are only generalized cylinders over circular helices from Euclidean perspective.*

Theorem 2.10. (Aydin and Mihai, 2016) *There does not exist an isotropic minimal Aminov surface in* $\mathbb{I}^4$.

Furthermore, same authors derived the following classification results for the spherical product surface $\left(M^2, c_1 \otimes c_2\right)$ of two curves c_1 and c_2 in $\mathbb{I}^4$ which is defined by

$$\mathbf{r} := c_1 \otimes c_2 : \mathbb{R}^2 \to \mathbb{I}^4,$$
$$(u,v) \mapsto (u, f_1(u), f_2(u)v, f_2(u)g(v)),$$

where the curves $c_1(u) = (u, f_1(u), f_2(u))$ and $c_2(v) = (v, g(v))$ are called *generating curves* of the surface.

Theorem 2.11. (Aydin and Mihai, 2016) *Let* $\left(M^2, c_1 \otimes c_2\right)$ *be a isotropic flat spherical product surface in* $\mathbb{I}^4$. *Then either it is a non-isotropic plane or one of the following satisfies*

(i) c_1 *is a planar curve in* $\mathbb{I}^3$ *lying in the non-isotropic plane* $z = const.$;

(ii) c_1 *is a line in* $\mathbb{I}^3$;

(iii) c_1 is *a curve in* $\mathbb{I}^3$ *of the form*

$$c_1(u) = \left(u, f_1(u), \lambda \int \sqrt{1 + \left(f_1'\right)^2}\, du + \xi\right),$$
$$\lambda, \xi \in \mathbb{R},\ \lambda \neq 0;$$

(iv) c_2 *is a line in* $\mathbb{I}^2$.

Theorem 2.12. (Aydin and Mihai, 2016) *There does not exist an isotropic minimal spherical product surface in* $\mathbb{I}^4$ *except totally geodesic ones.*

3. Helicoidal surfaces in $\mathbb{I}^3$

The rotation surfaces in the Euclidean 3-space $\mathbb{R}^3$ with constant mean curvature have been known for a long time Delaunay (1841), Kenmotsu (1980). A natural generalization of rotation surfaces in $\mathbb{R}^3$ are the *helicoidal surfaces* that can be defined as the orbit of a plane curve under a screw motion in $\mathbb{R}^3$.

Such surfaces in $\mathbb{R}^3$ with constant mean and constant Gaussian curvature have been classified by (Do Carmo and Dajczer, 1982). These classifications were extended to the ones with prescribed mean and Gaussian curvatures by (Baikoussis and Koufogiorgos, 1998)

The helicoidal surfaces also have been studied by many authors as focusing on curvature properties in the Minkowskian 3-space $\mathbb{R}^3_1$, the pseudo-Galilean space $\mathbb{G}^1_3$ and several homogeneous spaces, see (Arvanitoyeorgos and Kaimakamis, 2010), (Beneki et al., 2002) etc.

Morever, there exist many works related with the helicoidal surfaces satisfying an equation in terms of its position vector and Laplace operator in $\mathbb{R}^3$ and $\mathbb{R}^3_1$. For example see (Baba-Hamed and Bekkar, 2009), (Choi et al., 2010), etc.

Now we adapt the above notion to $\mathbb{I}^3$. Considering the i-motions given by (1.1), the Euclidean rotation in the isotropic space $\mathbb{I}^3$ is given by in the normal form (in affine coordinates)

$$\begin{cases} x_1' = x_1 \cos\phi - x_2 \sin\phi, \\ x_2' = x_1 \sin\phi + x_2 \cos\phi, \\ x_3' = x_3, \end{cases}$$

where $\phi \in \mathbb{R}$.

Now let c be a curve lying in the isotropic x_1x_3-plane given by $c(u) = (f(u), 0, g(u))$, where $f, g \in C^2$ and $f \neq 0 \neq \frac{df}{du}$. By rotating the curve c around $z-$ axis and simultaneously followed by a translation, we obtain that the *helicoidal surface of* *first type* in $\mathbb{I}^3$ with the profile curve c and pitch h is of the form

$$\begin{cases} \mathbf{r}(u,v) = (f(u)\cos v, f(u)\sin v, g(u)+hv) \\ h \in \mathbb{R}. \end{cases} \quad (3.1)$$

Similarly when the profile curve c lies in the isotropic x_2x_3-plane, then the *helicoidal surface of second type* in $\mathbb{I}^3$ with pitch h is given by

$$\begin{cases} \mathbf{r}(u,v) = (-f(u)\sin v, f(u)\cos v, g(u)+hv) \\ h \in \mathbb{R}. \end{cases} \quad (3.2)$$

In the particular case $h=0$, these reduce to the *surfaces of revolution* in $\mathbb{I}^3$. Also when g is a constant, then it is a *helicoid* from Euclidean perspective.

Remark 3.1. The coordinate functions f and g of the profile curve c of a helicoidal surface in $\mathbb{I}^3$ are arbitrary functions of class C^2 and so one can take $f(u)=u$.

Remark 3.2. Since both type of the helicoidal surfaces are locally isometric, we only will focus on the ones of first type.

Let M^2 be a helicoidal surface of first type in $\mathbb{I}^3$. Then the matrix of the first fundamental form a of M^2 is

$$(a_{ij}) = \begin{bmatrix} 1 & 0 \\ 0 & u^2 \end{bmatrix} \text{and } (a^{ij}) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{u^2} \end{bmatrix},$$

where $(a^{ij}) = (a_{ij})^{-1}$. Thus the Laplacian of M^2 with respect to a is

$$\Delta = \frac{1}{\sqrt{\det(a_{ij})}} \sum_{i,j=1}^{2} \frac{\partial}{\partial u_i}\left(\sqrt{\det(a_{ij})} a^{ij} \frac{\partial}{\partial u_j} \right)$$

and by taking $u_1 = u$ and $u_2 = v$, we get

$$\Delta = \frac{1}{u}\frac{\partial}{\partial u} + \frac{\partial^2}{\partial u^2} + \frac{1}{u^2}\frac{\partial^2}{\partial v^2}.$$

Putting $r_1(u,v) = u\cos v$, $r_2(u,v) = u\sin v$ and $r_3(u,v) = g(u) + hv$, one can easily seen that

$$\Delta r_1 = \Delta r_2 = 0 \text{ and } \Delta r_3 = \frac{1}{u}g' + g'',$$

where the prime denotes the derivative with respect to u. Assuming $\Delta r_3 = \lambda r_3$, $\lambda \in \mathbb{R}$, we can obtain that λ must be zero and

$$\frac{1}{u}g' + g'' = 0. \quad (3.3)$$

After solving (3.3), we derive $g(u) = \alpha \ln|u| + \beta$ for $\alpha \in \mathbb{R} \setminus \{0\}$, $\beta \in \mathbb{R}$.

Thus we have the following result

Proposition 3.3. *Let M^2 be a helicoidal surface of first type in $\mathbb{I}^3$ satisfying $\Delta r_i = \lambda_i r_i$, $\lambda_i \in \mathbb{R}$. Then it is isotropic minimal and has the form*

$$\mathbf{r}(u,v) = (u\cos v, u\sin v, \alpha \ln|u| + hv + \beta)$$

for $\alpha \in \mathbb{R} \setminus \{0\}, \beta \in \mathbb{R}$.

4. Helicoidal surfaces of constant curvature in $\mathbb{I}^3$

Let us consider the helicoidal surface of first type M^2 in $\mathbb{I}^3$. Then the components of the second fundamental form are

$$b_{11} = g'', \; b_{12} = -\frac{h}{u}, \; b_{22} = ug'. \quad (4.1)$$

Thereby, the relative curvature K of M^2 is

$$K = \frac{u^3 g' g'' - h^2}{u^4}. \quad (4.2)$$

Assume that M^2 has constant relative curvature K_0. We have to consider two cases:

Case (a). K vanishes. It follows from (4.2) that $g'g'' = \frac{h^2}{u^3}$ or

$$g'(u) = \left(\alpha - \frac{h^2}{u^2} \right)^{\frac{1}{2}}, \; \alpha \in \mathbb{R}^+. \quad (4.3)$$

After integrating (4.3), we obtain

$$g(u) = \sqrt{\alpha u^2 - h^2} + h\arctan\left(\frac{h}{\sqrt{\alpha u^2 - h^2}} \right).$$

Case (b). K is a nonzero constant K_0. Then we can rewrite (4.2) as

$$g'g'' = K_0 u + \frac{h^2}{u^3}$$

or

$$g'(u) = \left(K_0 u^2 - \frac{h^2}{u^2} + \gamma \right)^{\frac{1}{2}}, \; \gamma \in \mathbb{R}. \quad (4.4)$$

By integrating (4.4), we derive

$$g(u) = \frac{1}{4}\left(2d(u) - 2h\arctan\left(\frac{-2h^2 + \gamma u^2}{2hd(u)} \right) + \right.$$

$$\left. + \frac{\gamma}{\sqrt{K_0}} \ln\left| \gamma + 2(K_0 u^2 + \sqrt{K_0} d(u)) \right| \right),$$

where $\gamma \in \mathbb{R}$ and $d(u) = \sqrt{K_0 u^4 - h^2 + \gamma u^2}$.

Thus, we have the next result

Theorem 4.1. *Let M^2 be a helicoidal surface in I^3 with constant relative curvature K_0. Then we have the following items*

(i) when $K_0 = 0,\ M^2$ has the form

$$\begin{cases}\mathbf{r}(u,v) = (u\cos v, u\sin v, g(u) + hv), \\ g(u) = \sqrt{\alpha u^2 - h^2} + h\arctan\left(\frac{h}{\sqrt{\alpha u^2 - h^2}}\right), \\ \alpha \in \mathbb{R}^+, \end{cases} \quad (4.5)$$

(ii) *otherwise, i.e. $K_0 \neq 0$, it is of the form*

$$\begin{cases}\mathbf{r}(u,v) = (u\cos v, u\sin v, g(u) + hv), \\ g(u) = \dfrac{d(u)}{2} - \dfrac{h}{2}\arctan\left(\frac{-2h^2+\gamma u^2}{2hd(u)}\right) + \\ + \frac{\gamma}{4\sqrt{K_0}}\ln\left|\gamma + 2(K_0u^2 + \sqrt{K_0}d(u))\right|, \\ d(u) = \sqrt{K_0u^4 - h^2 + \gamma u^2},\ \gamma \in \mathbb{R}. \end{cases} \quad (4.6)$$

Example 4.2. Take $h = 1$, $\alpha = 1$, $u \in [1,5]$ and $v \in [0,4\pi]$ in (4.5). Then M^2 becomes isotropic flat and can be drawn as in Figure 1.

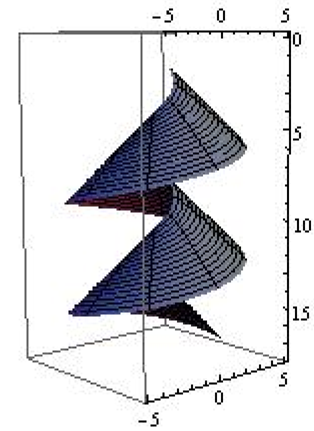

Figure 1. A helicoidal surface $K_0 = 0,\ h = 1$.

The isotropic mean curvature H of M^2 is given by

$$2H = \frac{g'}{u} + g''.$$

Suppose that M^2 has constant isotropic mean curvature H_0. Then putting $g' = p$, we obtain the following Riccati equation

$$p' + \frac{p}{u} = 2H_0. \quad (4.7)$$

Solving (4.7) we get

$$g(u) = \frac{H_0}{2}u^2 + \alpha\ln|u| + \beta$$

for some constants $\alpha, \beta \in \mathbb{R}$ and $\alpha \neq 0$.

Therefore we have proved the next result:

Theorem 4.3. *Let M^2 be a helicoidal surface in I^3 with constant isotropic mean curvature H_0. Then it has the following form*

$$\begin{cases}\mathbf{r}(u,v) = (u\cos v, u\sin v, g(u) + hv), \\ g(u) = \frac{H_0}{2}u^2 + \alpha\ln|u| + \beta,\ \alpha \in \mathbb{R} \setminus \{0\}. \end{cases} \quad (4.8)$$

Example 4.4. Let us put $h = 1.5$, $H_0 = -\alpha = -1$, $\beta = 0$, $u \in [1,5]$ and $v \in [-\pi,\pi]$ in (4.8). Then we draw it as in Figure 2.

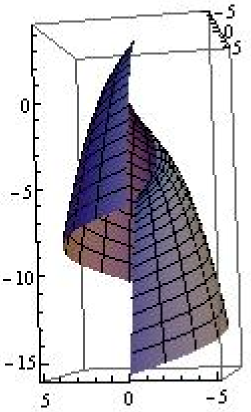

Figure 2. A helicoidal surface $H_0 = -1,\ h = 1.5$.

5. Special curves on the helicoidal surfaces in I^3

For more details of special curves on the surfaces in I^3 such as, geodesics, asymptotic lines and lines of curvature, see Sachs (1990b), p. 163-181.

In this section we aim to investigate such curves on a helicoidal surface in I^3.

Let M^2 be a helicoidal surface in I^3, then any point of a curve on M^2 has the position vector

$$\mathbf{r}(u(s),v(s)) = \mathbf{r}(s) = (u(s)\cos(v(s)), u(s)\sin(v(s)), g(u(s)) + hv(s)), \quad (5.1)$$

where s is arc-length parameter of $\mathbf{r}(s)$. Denote the derivative with respect to s by a dot. Then $\mathbf{t}(s) = \dot{\mathbf{r}}(s) = (t_1(s), t_2(s), t_3(s))$ is the tangent vector of $\mathbf{r}(s)$. We can take a side tangential vector $\sigma(s) = (\sigma_1(s), \sigma_2(s), \sigma_3(s))$ in the tangent plane of M^2 such that

$$\sigma_1^2 + \sigma_2^2 = 1,\ \sigma_1 t_1 + \sigma_2 t_2 = 0, t_1\sigma_2 - t_2\sigma_1 = 1.$$

Therefore we have an orthonormal triple $\{\mathbf{t}, \sigma, \mathbf{N} = (0,0,1)\}$. The second derivative of $\mathbf{r}(s)$ with respect to s has the following decomposition

$$\ddot{\mathbf{r}} = \kappa_g \sigma + \kappa_n \mathbf{N},$$

where κ_g and κ_n are respectively called the *geodesic curvature* and *normal curvature* of $\mathbf{r}(s)$ on M^2. The curve $\mathbf{r}(s)$ is called *geodesic* (resp., *asymptotic line*) if and only if its geodesic curvature κ_g (resp., normal curvature κ_n) vanishes.

The first derivative of $\sigma(s)$ with respect to s has the decomposition

$$\dot{\sigma} = -\kappa_g \mathbf{t} + \tau_g \mathbf{N},$$

in which τ_g is called the *geodesic torsion* of $\mathbf{r}(s)$ on M^2.

In terms of the components of the first fundamental form of M^2, the side tangential vector σ is given by

$$\sigma = -\frac{1}{\sqrt{\det(a_{ij})}}\left[(a_{12}\dot{u} + a_{22}\dot{v})\mathbf{r}_u - (a_{11}\dot{u} + a_{12}\dot{v})\mathbf{r}_v\right].$$

So, the geodesic curvature of $\mathbf{r}(s)$ on M^2 in $\mathbb{I}^3$ is given by

$$\kappa_g(s) = u^2(\dot{v})^3 - u\dot{u}\ddot{v} - 2(\dot{u})^2\dot{v} - u\dot{v}\ddot{u}. \quad (5.2)$$

It is easliy seen from (5.2) that the curves $v = const.$ on M^2 are geodesics of M^2 but not the curves $u = const.$, which implies the next result.

Theorem 5.1. *The $v-$parameter curves on the helicoidal surfaces in $\mathbb{I}^3$ are geodesics but not $u-$parameter curves.*

The normal curvature of $\mathbf{r}(s)$ on M^2 in $\mathbb{I}^3$ is

$$\kappa_n(s) = g''(\dot{u})^2 - 2\frac{h}{u}(\dot{u}\dot{v}) + ug'(\dot{v})^2. \quad (5.3)$$

By (5.3) the curves $u = const.$ are asymptotic lines of M^2 if and only if g is a constant function. Similarly the curves $v = const.$ are asymptotic lines of M^2 if and only if g is a linear function.

Hence, we have proved the following

Theorem 5.2. *(i) The $u-$parameter curves on a helicoidal surface in $\mathbb{I}^3$ are asymptotic curves if and only if it is a helicoid from Euclidean perspective;*
(ii) the $v-$parameter curves on the helicoidal surfaces in $\mathbb{I}^3$ are asymptotic curves if and only if g is a linear function.

On the other hand a curve on a surface is called a *line of curvature* if its geodesic torsion τ_g vanishes. The function τ_g can be defined as

$$\tau_g = \frac{\begin{vmatrix} dv^2 & -dudv & du^2 \\ a_{11} & a_{12} & a_{22} \\ b_{11} & b_{12} & b_{22} \end{vmatrix}}{\det(a_{ij})a}.$$

Hence, a curve on M^2 in $\mathbb{I}^3$ is a line of curvature if and only if the following equation satisfies

$$-\left(\frac{h}{u}\right)(\dot{u})^2 + (ug' - u^2 g'')\dot{u}\dot{v} + (hu)(\dot{v})^2 = 0.$$

Therefore we can give the following result.

Theorem 5.3. *The parameter curves on the helicoidal surfaces in $\mathbb{I}^3$ are lines of curvature if and only if these are surfaces of revolution.*

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